Mathematical Logic IV

The Lambda Calculus; **by H.P. Barendregt**(1984)

Part One: Chapters 1-5

1 Introduction

The λ-calculus (a theory denoted $\boldsymbol{\lambda}$) is a type free theory about functions as *rules*, rather that as graphs {i.e. sets of n-tuples} in order to stress their *computational* aspects. This way of thinking about functions (as rules) involves the basic idea of a function where given certain input the function will, in some sense, generate an output. It seems to me that this is our intutive notion of a function, where a function is a specified "way" of getting something (output) from something (input) as opposed to merely being a set of n-tuples.

A function is a rule of correspondence by which when anything is given (as argument) another thing (the value of the function for that argument) may be obtained. That is, a function is an operation which may be applied on one thing (the argument) to yield another thing (the value of the function). [1]

Application is a primitive operation in λ-calculus. The function f *applied* to a is denoted by fa.

Abstraction is a sort of generalization over functions. Let $t(\equiv t(x))$ be an expression possibly containing x. Then $\lambda x.t(x)$ is the function f that assigns to the argument a the value $t(a)$. That is,

$$(\beta) \qquad (\lambda x.t(x))a = t(a)$$

Think of this as something like the following: $\lambda x.t(x)$ is the function that applies a given argument (in this case a) to the function t only and yields the value $t(a)$ (i.e., It "substitutes" a in for all of the bound x's in t).

The theory $\boldsymbol{\lambda}$ has as terms the set Λ ($\lambda - terms$) built up from variables using application and abstraction. The statements of $\boldsymbol{\lambda}$ are equations between the terms in Λ. Also $\boldsymbol{\lambda}$ has as its only mathematical axioms the scheme (β).

2 Conversion

2.1 λ-terms and conversion

The principle object of study in the λ-calculus is the set of lambda terms modulo convertibility***.

2.1.1 (Definition) (i) *Lambda terms* are words over the following alphabet:

$v_0, v_1, \ldots$	variables
λ	abstractor
(,)	parentheses

(ii) The set of λ-terms, Λ, is defined inductively as follows:

(1) $x \in \Lambda$, x is arbitrary
(2) $M \in \Lambda \Rightarrow (\lambda x M) \in \Lambda$, x is arbitrary
(3) $M,\ N \in \Lambda \Rightarrow (MN) \in \Lambda$

2.1.2 (Notation) (i) $M,\ N,\ L,\ \ldots$ denote arbitrary λ-terms.
(ii) $x,\ y,\ z,\ \ldots$ denote arbitrary variables.
(iii) Outermost parentheses are not written.
(iv) The symbol $\equiv$ denotes syntactic equality (i.e., have the same form).

2.1.3 (more Notation) (i) Let $\vec{x} \equiv x_1, \ldots, x_n$.

[1]A. Church *The Calculi of Lambda-Conversion*-1941-Princeton University Press

$$\text{Then } \lambda x_1 \cdots x_n.M \equiv \lambda \vec{x}.M \equiv \lambda x_1(\lambda x_2(\cdots(\lambda x_n(M))\ldots))$$

Let $\vec{N} \equiv N_1, \ldots, N_n$.

$$\text{Then } MN_1 \cdots N_n \equiv M\vec{N} \equiv (\cdots((MN_1)N_2)\cdots N_n)$$
(association to the left)

(iii) $\| M \|$ is the length of M (i.e., the number of symbols in M).

The *substitution operator*,

$M[x := N]$ denotes the result of substituting N for x in M.

The basic eqivalence relation on λ-terms is that of *convertibility*, which will be generated by axioms.

2.1.4 (Definition) The theory $\boldsymbol{\lambda}$ has as formulas

$$M = N$$

where $M, N \in \Lambda$ and is axiomatized by the following axioms and rules:

(I)	$(\lambda x.M)N = M[x := N]$,	(β-conversion)
(II.1)	$M = M$	
(II.2)	$M = N \Rightarrow N = M$	
(II.3)	$M = N, N = L \Rightarrow M = L$	
(II.4)	$M = N \Rightarrow MZ = NZ$	
(II.5)	$M = N \Rightarrow ZM = ZN$	
(II.6)	$M = N \Rightarrow \lambda x.M = \lambda x.N$	(rule ξ)

Provability in $\boldsymbol{\lambda}$ of an equation is denoted by $\boldsymbol{\lambda} \vdash M = N$ or just by $M = N$. If $\boldsymbol{\lambda} \vdash M = N$, then M and N are called *convertible*.

Note: $\boldsymbol{\lambda}$ is logic free: it merely consists of equations. Connectives and quantifiers will be used in the informal metalanguage discussing about $\boldsymbol{\lambda}$.

2.1.6 (Definition) A variable x occurs *free* in a λ-term M if x is not in the scope of λx; x occurs *bound* otherwise. This is the same binding criteria of the usual quanifiers.

2.1.7 (Definition) (i) $\mathbf{FV}(M)$ is the set of free variables in M and can be inductively defined as follows:

$\mathbf{FV}(x) = \{x\}$
$\mathbf{FV}\ (\lambda x.M) = \mathbf{FV}(M) - \{x\}$
$\mathbf{FV}(MN) = \mathbf{FV}(M) \cup \mathbf{FV}(N)$

(ii)M is *closed* or a *combinator* if $\mathbf{FV}(M) = \emptyset$.
(iii) $\Lambda^0 = \{M \in \Lambda \mid M \text{ is closed}\}$
(iv) $\Lambda^0(\vec{x}) = \{M \in \Lambda \mid \mathbf{FV}(M) \subseteq \{\vec{x}\}\}$
(v) A *closure* of $M \in \Lambda$ is $\lambda \vec{x}.M$, where $\{\vec{x}\} = \mathbf{FV}(M)$
– Note that a closure of M depends upon the order of the $\vec{x}$.

2.1.8 (Defintion) (i) M is *subterm* of N (denoted by $M \subset N$) if $M \in \text{Sub}(N)$, where $\text{Sub}(N)$, the collection of subterms of N, is defined inductively as follows:
$\text{Sub}(x) = \{x\}$
$\text{Sub}(\lambda x.N_1) = \text{Sub}(N_1) \cup \{\lambda x.N_1\}$
$\text{Sub}(N_1N_2) = \text{Sub}(N_1) \cup \text{Sub}(N_2) \cup \{N_1N_2\}$

Subterms may occur several times and two subterms are *disjoint* if they have nome common symbol occurances. A subterm N of M is *active* if N occurs as $(NZ) \subset M$ for some Z; otherwise N is *passive*.

2.1.9 (Definition) Let $F, M \in \Lambda$. Then

(i) $F^0M \equiv M; F^{n+1}M \equiv F(F^nM)$

(ii) $FM^{\sim 0} \equiv F; FM^{\sim n+1} \equiv FM^{\sim n}M$

2.1.11 (Definition) (i) A *change of bound variables* in M is the replacement of a part $\lambda x.N$ of M by $\lambda y.(N[x := y])$, where y does not occur (at all) in N. Since y is a fresh variable there is no threat of messing up the binding properties of the term.
(ii) M is α-*congruent* with N (denoted by $M \equiv_a N$) if N results from M by a series of changes of bound variables. [2] **2.1.12** (Convention) Terms that are α-conguent are identified. So now we can simply write $\lambda x.x \equiv \lambda y.y$, etc.

2.1.13 (variable Convention) If $M_1, \ldots, M_n$ occur in a certain mathematical context (e.g. definition, proof, etc), then in these terms all bound variables are chosen to be different from the free variables.

2.1.15 (Definition) The result of *substituting* N for the free occurrences of x in M (denoted by $M[x := N]$) is defined as follows:

$$x[x := N] \equiv N$$

$$y[x := N] \equiv y, \text{ if } x \not\equiv y$$

$$(\lambda y.M_1)[x := N] \equiv \lambda y.(M_1[x := N])$$

$$(M_1M_2)[x := N] \equiv (M_1[x := N])(M_2[x := N])$$

2.1.16 (Substitution Lemma) If $x \neq y$ and $x \notin \mathbf{FV}(L)$, then

$$M[x := N][y := L] \equiv M[y := L][x := N[y := L]]$$

Proof:

2.1.18 (Defintion) (i) A *context* $C[\ ]$ is a term with some holes in it. More formally:

x is a context
$[\ ]$ is a context
if $C_1[\ ]$ and $C_2[\ ]$ are contexts then so are $C_1[\ ]C_2[\ ]$ and $\lambda x.C_1[\ ]$

(ii) If $C[\ ]$ is a context and $M \in \Lambda$, then $C[M]$ denotes the result of placing M in the holes of $C[\ ]$. In this act free variables in M may become bound by $C[M]$.

2.1.21 (Defintion) (i) Let $\vec{N} \equiv N_1, \ldots, N_m; \vec{x} \equiv x_1, \ldots, x_n$. Then $\vec{N}$ *fits in* $\vec{x}$ if $m = n$ and the $\vec{x}$ do not occur in $\mathbf{FV}(\vec{N})$.

(ii) Let $\vec{N} \equiv N_1, \ldots, N_M; \vec{L} \equiv L_1, \ldots, L_n$. Then

$$\vec{N} = \vec{L} \text{ if } n = m \text{ and } N_i = L_i \text{ for } 1 \leq i \leq n$$

$\vec{N} \equiv \vec{L}$ is defined similarly.

(iii) Let $\vec{N}$ fit in $\vec{x} \equiv x_1, \ldots, x_n$. Then

$$M[\vec{x} := \vec{N}] \equiv M[x_1 := N_1], \ldots, M[x_n := N_n], \text{ where } \vec{x} \text{ do not occur in } \mathbf{FV}(\vec{N})$$

(iv) Let $M \in \Lambda$. As in predicate logic sometimes we write $M \equiv M(\vec{x})$, to indicate substitution:
if $M \equiv M(\vec{x})$ and $\vec{N}$ fits in $\vec{x}$, then $M(\vec{N}) \equiv M[\vec{x} := \vec{N}]$

2.1.22 (Lemma) Let $\vec{x} \equiv x_1, \ldots, x_n$. Then $(\lambda \vec{x}.M)\vec{x} = M$.

[2] This is similar to Church's α-conversion which said $\lambda x.M = \lambda y.M[x := y]$. The difference is that α conversion is a semantic notion (notice the use of '=') and α-congruity is a syntactic notion.

Proof:

2.1.24 (Combinatory Completeness) Let $M \equiv M(\vec{x})$. Then:

(i)$\exists F \;\; F\vec{x} = M(\vec{x})$
(ii)$\exists F \;\; \forall \vec{N} \;\; F(\vec{N}) = M(\vec{N})$, where $\vec{N}$ fits in $\vec{x}$
(iii) In (i), (ii) one can take $F \equiv \lambda x.M$

2.1.25 (Definition)

$$\mathbf{I} \equiv \lambda x.x,\ \mathbf{K} \equiv \lambda xy.x,\ \mathbf{S} \equiv \lambda xyz.xz(yz)$$

2.1.26 (Corollary) For all $M, N, L \in \Lambda$

(i) $\mathbf{I}M = M$
(ii) $\mathbf{K}MN = M$
(iii) $\mathbf{S}MNL = ML(NL)$

Lambda terms denote processes. Different terms may denote the same process.

2.1.27 (Definition) (i) *Extensionality* is the following derivation rule:

$$Mx = Nx \Rightarrow M = N \qquad (ext)$$

provided that $x \notin \mathbf{FV}(MN)$.

(ii) The theory $\boldsymbol{\lambda}$ extended by this rule is denoted by $\boldsymbol{\lambda}+ext$.

The only difference between $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}+ext$ is that $\boldsymbol{\lambda x.Mx = M}$ is provable in $\boldsymbol{\lambda}+ext$ and not in $\boldsymbol{\lambda}$.

2.1.28 (Definition) Consider the folloing axiom scheme $\boldsymbol{\eta}$

$$\boldsymbol{\lambda x.Mx = M} \qquad (\boldsymbol{\eta}\text{-conversion})$$

provided that $\boldsymbol{x} \notin \mathbf{FV}(\boldsymbol{M})$. $\boldsymbol{\lambda\eta}$ is the theory $\boldsymbol{\lambda}$ extended with $\boldsymbol{\eta}$.

2.1.29 (Theorem) The theories $\boldsymbol{\lambda}+ext$ and $\boldsymbol{\lambda\eta}$ are equivalent.[3]

2.1.30 (Definition) (i) An *equation* is a formula of the form $\boldsymbol{M = N}$ with $\boldsymbol{M, N} \in \boldsymbol{\Lambda}$; the equation is closed if $\boldsymbol{M, N} \in \boldsymbol{\Lambda^0}$.

(ii) Let $\mathscr{T}$ be a formal theory with equations as formulas. Then $\mathscr{T}$ is *consistent* (denoted by Con($\mathscr{T}$)) if $\mathscr{T}$ does not prove every closed equation. If $\mathscr{T}$ does prove every closed equation then it is *inconsistent*.

(iii) If $\mathscr{T}$ is a set of equations, then $\boldsymbol{\lambda}+\mathscr{T}$ is the theory obtained from $\boldsymbol{\lambda}$ by adding the equations of $\mathscr{T}$ as axioms. $\mathscr{T}$ is called *consistent* if Con($\boldsymbol{\lambda}+\mathscr{T}$).

2.1.31 (Fact) The theories $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda\eta}$ are consistent.[4]

2.1.32 (Definition) Let $\boldsymbol{M, N} \in \boldsymbol{\Lambda}$. Then $\boldsymbol{M}$ and $\boldsymbol{N}$ are *incompatible* (denoted by $\boldsymbol{M \# N}$), if $\neg$Con($\boldsymbol{M = N}$).

[3]Theorem **2.1.29** is attributed to Curry. Also, the extensional $\boldsymbol{\lambda}$-calculus is usually denoted by $\boldsymbol{\lambda\eta}$.
[4]The theory $\boldsymbol{\lambda}$ extended by a single axiom may become inconsistent.

Normal Forms
Consider a term like

$$(\lambda x.xa)\mathbf{I}$$

This term can be computed to yield

$$\mathbf{I}a$$

and gives

$$a$$

The term a is called a normal form, for it does not "compute" any further.

2.1.34 (Definition) Let $M \in \Lambda$.

(i) M is a β*-normal form* (abbreviated by β-nf or just nf) if M has no subterm $(\lambda x.R)L$

(ii) M has a β-nf if there exists an N such that $N = M$ and N is a β-nf.[5]

2.1.35 (Definition) Let $M \in \Lambda$.

(i) M is a $\beta\eta$-nf if M has no subterm $(\lambda x.P)Q$ or $(\lambda x.Rx)$ with $x \notin \mathbf{FV}(R)$.

(ii) M has a $\beta\eta$-nf if

$$\exists N[\lambda\eta \vdash M = N \text{ and } N \text{ is a } \beta\eta\text{-nf.}]$$

2.1.36 (Fact) M has a $\beta\eta$-nf $\Leftrightarrow$ M has a β-nf.[6]

2.1.37 (Fact) (i) If M, N are different β-nf's, then

$$\lambda \nvdash M = N$$

(ii) Similarly for $\beta\eta$-nf's and provability in $\lambda\eta$.

2.1.38 (Corollary) The Theories λ and $\lambda\eta$ are consistent.

2.1.39 (Fact) If M, N are different $\beta\eta$-nf's, then $M\#N$.[7]

2.1.40 (Theorem) Suppose M, N have a nf. Then either $\lambda\eta\vdash M = N$ or $\lambda\eta + M = N$ is inconsisten.

3 Reduction

There is a certain asymmetry in the defining equation for λ-abstraction. The statement

$$(\lambda x.x^2 + 1)3 = 10$$

can be interpreted as "10 is the result of computing $(\lambda x.x^2 + 1)3$" (i.e., 10 is the result of the function applying 3 to the function $x^2 + 1$). This computational aspect will be expressed by writing

$$(\lambda x.x^2 + 1)3 \ \rightarrow\ 10$$

which reads "$(\lambda x.x^2 + 1)3$ *reduces to* 10."

[5]If M is a nf, then it is also said to be *in* nf.
[6]Curry et al. [1972]
[7]Bőhm [1968]

The Church-Rosser theorem says that if two terms are convertible, then there is a term to which they both reduce. In many cases the inconvertibility of two terms can be proved by showing that they don't reduce to a common term.

3.1 Notions of Reduction

3.1.1 (Definition) (i) A binary relation $\boldsymbol{R}$ on $\boldsymbol{\Lambda}$ is *compatible* (with the operations) if

$$(M, M') \in R \Rightarrow (ZM, ZM') \in R,\ (MZ, M'Z) \in R \text{ and } (\lambda x.M, \lambda x.M') \in R$$

for all $M, M', Z \in \Lambda$.
(ii) An *equality* (or *congruence*) *relation* on Λ is a compatible equivalence relation.
(iii) A *reduction relation* on Λ is one which is compatible, reflexive, and transitive.

Note: A relation $R \subseteq \Lambda^2$ is compatible if

$$(M, M') \in R \Rightarrow (C[M], C[M']) \in R$$

for all $M, M' \in \Lambda$ and all contexts $C[\ ]$, with one hole.

3.1.2 (Definition) (i) A *notion of reduction* on Λ is a just a binary relation R on Λ.
(ii) If R_1, R_2 are notions of reduction, then $R_1 R_2$ is $R_1 \cup R_2$.

3.1.3 (Definition) $\beta = \{((\lambda x.M)N, M[x := N]) \mid M, N \in \Lambda\}$.

3.1.4 (Definition) If $\succ$ is a binary relation on a set X, then the *reflexive closure* of $\succ$ (denoted by $\succeq$) is the least relation extending R that is reflexive. The *transitive closure* (denoted by $\succ^*$) and the *compatible closure* are defined similarly.

3.1.5 (Defintion) Let R be a notion of reduction on Λ.
(i) Then R induces the binary realations

$\rightarrow_R$ *one step R-reduction*
$\twoheadrightarrow_R$ *R-reduction*
$=_R$ *R-equality* (also called *R-convertibility*)

inductively defined as follows. $\rightarrow_R$ is the compatible closure of R:

(1) $(M, N) \in R \Rightarrow M \rightarrow_R N$
(2) $M \rightarrow_R N \ \Rightarrow ZM \rightarrow_R ZN$
(3) $M \rightarrow_R N \ \Rightarrow MZ \rightarrow_R NZ$
(4) $M \rightarrow_R N \ \Rightarrow \lambda x.M \rightarrow_R \lambda x.N$

$\twoheadrightarrow_R$ is the reflexive, transitive closure of $\rightarrow_R$:

(1) $M \rightarrow_R N \ \Rightarrow M \twoheadrightarrow_R N$
(2) $M \twoheadrightarrow_R M$
(3) $M \twoheadrightarrow_R N, N \twoheadrightarrow_R L \ \Rightarrow M \twoheadrightarrow_R L$

$=_R$ is the equivalence relation generated by $\twoheadrightarrow_R$:

(1) $M \twoheadrightarrow_R N \ \Rightarrow M =_R N$
(2) $M =_R N \ \Rightarrow N =_R M$
(3) $M =_R N, N =_R L \ \Rightarrow M =_R L$

(ii) The basic relations derived from R are pronounced as follows:

$M \twoheadrightarrow_R N$: M *R-reduces to* N or N is an *R-reduct* of M
$M \rightarrow_R N$: M *R-reduces to* N *in one step*
$M =_R N$: M *is R-convertible to* N

The relations $\rightarrow_R, \twoheadrightarrow_R$, and $=_R$ are introduced inductively. Therefore properties about these relations can be proved inductively.

3.1.6 (Lemma) The relations $\rightarrow_R, \twoheadrightarrow_R$, and $=_R$ are all compatible. Therefore, $\twoheadrightarrow_R$ is a reduction relation and $=_R$ is an equality relation.
Proof:

3.1.7 (Remarks) By the compatibility of $\twoheadrightarrow_R$ it follows (by induction on the structure M) that

$$N \twoheadrightarrow_R N' \Rightarrow M[x := N] \twoheadrightarrow_R M[x := N']$$

(ii) The notion of compatible relation can be generalized directly to any set X with some operations on it. We can then speak of equality and reduction relations.

(iii) Notions of reduction will be denoted by boldface letters; e.g. $\boldsymbol{\beta}, \boldsymbol{\eta}, \boldsymbol{\Omega}$. The derived relations will be written using the corresponding lightface symbols; e.g. $\rightarrow_\beta, \twoheadrightarrow_\beta$, etc.

For the remainder of the section R is a notion of reduction on Λ.

3.1.8 (Definition) (i) An *R-redex* is a term M such that $(M, N) \in R$ for some term N. In this case N is called an *R-contractum* of M.

(ii) A term M is called an *R-normal form* (*R-nf*) if M does not contain (as subterm) any *R-redex*.

(iii) A term N is an *R-nf of M* (or *M has the R-nf N*) if N is an *R-nf* and $M =_R N$.

The process of stepping from a redex to a contractum is called *contraction*.

3.1.9 (Lemma)

3.1.10 (Corollary)

3.1.11 (Definition) (i) Let $\succ$ be a binary relation on Λ. Then $\succ$ satisfies the *diamond property* (denoted by $\succ \models \diamondsuit$) if

$$\forall M, M_1, M_2\ [M \succ M_1\ \wedge\ M \succ M_2 \Rightarrow \exists M_3[M_1 \succ M_3\ \wedge\ M_2 \succ M_3]]$$

see figure 3.1 (p. 54)

(iii) A notion of reduction R is said to be *Church-Rosser* (CR) if $\twoheadrightarrow_R$ satisfies the diamond property.

3.1.12 (Theorem)

3.1.13 (Corollary)

3.1.14 (Definition) A binary relation R on Λ is *substitutive* if for all $M, N, L \in \Lambda$ and all variables x one has

$$(M, N) \in R \Rightarrow (M[x := L], N[x := L]) \in R$$

3.1.15 (Proposition)

3.1.16 (Proposition)

3.1.17 (Definition) (i) Let Δ be a subterm occurrence of M, that is, $M \equiv C[\Delta]$. Write

$$M \xrightarrow{\Delta}_R N$$

if Δ is an R-redex with contractum Δ' and $N \equiv C[\Delta']$.

(ii) A *R-reduction* (path) is a finite or infinite sequence

$$M_0 \xrightarrow{\Delta_0}_R M_1 \xrightarrow{\Delta_1}_R M_2 \rightarrow_R \dots$$

3.1.18 (Conventions) (i) $\sigma,\ \tau, \dots$ range over reduction paths.

(ii) The reduction path σ in definition 3.1.17(ii) starts with M_0. If there is a last term M_n in σ, the σ ***ends*** with M_n. In that case one also says that σ is a reduction path from M_0 to M_n. If $n = 0$, then σ is called the *empty reduction* (denoted by $\emptyset :\ M_0 \twoheadrightarrow_R M_0$). If $n \neq 0$, then σ is a *proper* R-reduction (denoted by $(\sigma :)\ M_0 \underset{\neq\emptyset}{\twoheadrightarrow}_R M_n$.

(iii) Sometimes the Δ_0, Δ_1 are left out in denoting a reduction path.

(iv) We often write $\sigma : M_0 \rightarrow M_1 \rightarrow \dots$ to indicate that σ ***is*** the path $M_0 \rightarrow M_1 \rightarrow \dots$

(v) If $\sigma : M_0 \rightarrow \dots \rightarrow M_n$ and $\tau : M_n \rightarrow M_m$, then

$$\sigma + \tau : M_0 \rightarrow \dots \rightarrow M_n \rightarrow \dots \rightarrow M_m$$

(vi) If Δ is an R-redux occurrence in M with contractum Δ', then (Δ) denotes the one step reduction $M \xrightarrow{\Delta}_R N$. That is,

$$(\Delta) : C[\Delta] \xrightarrow{\Delta}_R C[\Delta']$$

(vii) If σ is a R-reduction path, then $\parallel \sigma \parallel$ is its *length* (i.e., the number of $\rightarrow_R$ steps in it). Note that $\parallel \sigma \parallel \in \mathbb{N} \cup \{\infty\}$.

3.1.20 (Definition) The R(reduction) *graph* of a term M (denoted by $G_R(M)$) is the set

$$\{N \in \Lambda \mid M \twoheadrightarrow_R N\}$$

directed by $\rightarrow_R$: if several redexes give rise to $M_0 \rightarrow_R M_1$, then that many directed arcs connect M_0 to M_1 in $G_R(M)$.

3.1.22 (Definition) Let $M \in \Lambda$.

(i) M R-*strongly normalizes* (denoted by R-SN(M)) if there is no infinite R-reduction starting with M.

(ii) M is R-*infinite* (denoted by $R-\infty(M)$) if not R-SN(M).

(iii) R is *strongly normalizing* (SN) if $\forall M \in \Lambda$ R-SN(M).

3.1.23 (Fact)

3.1.24 (Definition) (i) A binary relation $\succ$ (on a set X) satisfies the *weak diamond property* if

$$\forall x, x_1, x_2[x \succ x_1 \wedge x \succ x_2 \Rightarrow \exists x_3[x_1 \overset{*}{\succeq} x_3 \wedge x_2 \overset{*}{\succeq} x_3]]$$

where $\overset{*}{\succeq}$ is the transitive reflexive closure of $\succ$.

(ii) A notion of reduction R is *weakly Church-Rosser* (WRC) if $\rightarrow_R$ satisfies the weak diamond property.

3.1.25 (Proposition)

3.1.26 (Notation) (i) R-NF $= \{M \in \Lambda \mid M$ is in R-nf$\}$, R-NF$^0 = R$-NF $\cap \Lambda^0$.

(ii) If $\mathfrak{X} \subseteq \Lambda$, then $M \in_R \mathfrak{X}$ iff $M' =_R M$ for some $M' \in \mathfrak{X}$.
In this notation, $M \in_\beta \beta$-NF iff M has a β-nf.

3.1.27 (Definition) (i) $M \in \Lambda^0$ is R-*solvable* if $\exists \vec{P} \in \Lambda$ $M\vec{P} =_R I$.

(ii) $M \in \Lambda$ is R-*solvable* if some closed substitution instance of M is R-solvable.

3.2 Beta Reduction

3.2.1 (Proposition)

3.2.2 (Lemma)

3.2.3 (Definition) Define a binary relation $\twoheadrightarrow_1$ on Λ inductively as follows:

$$M \twoheadrightarrow_1 M$$

$$M \twoheadrightarrow_1 M' \Rightarrow \lambda x.M \twoheadrightarrow_1 \lambda x.M'$$

$$M \twoheadrightarrow_1 M', N \twoheadrightarrow_1 N' \Rightarrow MN \twoheadrightarrow_1 M'N'$$

$$M \underset{1}{\twoheadrightarrow} M', N \underset{1}{\twoheadrightarrow} N' \Rightarrow (\lambda x.M)N \underset{1}{\twoheadrightarrow} M'[x := N']$$

3.2.4 (Lemma)

3.2.5 (Lemma)

3.2.6 (Lemma)

3.2.7 (Lemma)

3.2.8 (Theorem) The Church-Rosser Theorem (CR)

(i) β is CR.

(ii) $M =_\beta N \Rightarrow \exists Z[M \twoheadrightarrow_\beta Z \wedge N \twoheadrightarrow_\beta Z]$

3.2.9 (Corollary)

3.2.10 (Theorem)

3.2.11 (Convention) The notion of reduction β will be used throughout this book. Therefore, to simplify notation the subscripts will be suppressed. That is

$$\rightarrow_\beta,\ \twoheadrightarrow_\beta,\ G_\beta(M),\ \beta\text{-NF},\ \beta - \infty(M) \text{ and } \beta\text{-solvable}$$

will be denoted by

$$\rightarrow,\ \twoheadrightarrow,\ G(M), \text{ NF}, \infty(M) \text{ and solvable.}$$

Note: The notation $\in_\beta$ will NOT be replaced by $\in$.

3.3 Eta Reduction

3.3.1 (Definition) (i) η:$\lambda x.Mx \rightarrow M$ provided $x \notin \text{FV}(M)$; that is $\eta = \{(\lambda x.Mx, M) \mid x \notin \text{FV}(M)\}$

(ii) $\beta\eta = \beta \cup \eta$

The point of $\beta\eta$-reduction is that it axiomatizes provable equality in the extensional λ-calculus and it is CR.

3.3.2 (Proposition)

3.3.3 (Proposition)

3.3.4 (Definition) Let $\succ_1$ and $\succ_2$ be two binary relations on the set X. Then $\succ_1$ and $\succ_2$ *commute* if

$$\forall x, x_1, x_2 \in X[x \succ_1 x_1 \wedge x \succ_2 x_2 \Rightarrow \exists x_3[x_1 \succ_2 x_3 \wedge x_2 \succ_1 x_3]]$$

see figure 3.6 (page 64)

Note: $\succ \models \Diamond$ iff $\succ$ commutes with itself.

3.3.5 (Proposition)

3.3.6 (Lemma)

3.3.7 (Lemma)

3.3.8 (Lemma)

3.3.9 (Theorem)

3.3.10 (Corollary)

3.3.11 (Theorem)

3.3.12 (Proposition)

4 Theories

4.1 Lambda Theories

Lambda theories are consistent extensions of the $\boldsymbol{\lambda}$-calculus that are closed under derivations. They are studied because of their own interest and because of their application to ordinary $\boldsymbol{\lambda}$-conversion.

Remember that a (closed) equation is a formula of the form M=N, with $\boldsymbol{M, N} \in \boldsymbol{\Lambda^0}$. If $\mathscr{T}$ is a set of equations, then the theory $\boldsymbol{\lambda}\mathscr{T}$ is obtained by adding to the axioms and rules of the $\boldsymbol{\lambda}$-calculus the equations in $\mathscr{T}$ as new axioms.

4.1.1 (Definition) Let $\mathscr{T}$ be a set of closed equations.
$\mathscr{T}^+$ is the set of closed equations provable in $\boldsymbol{\lambda}+\mathscr{T}$.
$\mathscr{T}$ is a $\boldsymbol{\lambda}$-*theory* if $\mathscr{T}$ is consistent and $\mathscr{T}^+ = \mathscr{T}$.

By corollary 2.1.38 both $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda\eta}$ are $\boldsymbol{\lambda}$-theories.

4.1.2 (Remarks) (i) Since the rules $\boldsymbol{\xi}$ is in $\boldsymbol{\lambda}$, each $\boldsymbol{\lambda}$-theory $\mathscr{T}$ is closed under $\boldsymbol{\xi}$ and hence
$\mathscr{T} \vdash \boldsymbol{M = N} \Leftrightarrow \mathscr{T} \vdash \boldsymbol{\lambda x.M = \lambda x.N}$.
The $\Leftarrow$ follows since $\boldsymbol{(\lambda x.M)x = M}$ in $\mathscr{T}$.

(ii) By (i) it follows that it does not matter to resrict ourselves in 4.1.1 to sets of closed equations.

(iii) Clearly Con($\mathscr{T}$)$\Leftrightarrow$ $\boldsymbol{\lambda} + \mathscr{T} \nvdash \mathbf{T} = \mathbf{F}$.

(iv) Each $\boldsymbol{\lambda}$-theory is identified with the set of closed equations provable in it. In particular,
$\boldsymbol{\lambda} = \{\boldsymbol{M = N} | \boldsymbol{M, N} \in \boldsymbol{\Lambda^0}$ and $\boldsymbol{\lambda} \vdash \boldsymbol{M = N}\}$.

4.1.3 (Proposition) Let $\mathscr{T}$ be a $\boldsymbol{\lambda}$-theory. Then
(i) $\mathscr{T} \vdash \boldsymbol{M = M'} \Rightarrow \mathscr{T} \vdash \mathrm{C}[\boldsymbol{M}] = \mathrm{C}[\boldsymbol{M'}]$
(ii) $\mathscr{T}\boldsymbol{M = M'}, \mathscr{T} \vdash \boldsymbol{N = N'} \Rightarrow \mathscr{T} \vdash \boldsymbol{M[x := N] = M'[x := N']}$

Proof: (i) By induction on the structure of C[ ].

(ii) Assume $\mathscr{T} \vdash \boldsymbol{M = M'}$. Then by (i) $\mathscr{T} \vdash \boldsymbol{(\lambda x.M)N = \lambda.M')N}$ hence
$\mathscr{T} \vdash \boldsymbol{M[x := N] = M'[x := N']}$.

If moreover $\mathscr{T} \vdash \boldsymbol{N = N'}$, then by (i) $\mathscr{T} \vdash \boldsymbol{M'[x := N] = M'[x := N']}$ and we're done.

4.1.4 (Notation) Let $\mathscr{T}$ be a theory.
(i) $\mathscr{T} \vdash \boldsymbol{M = N}$ stands for $\boldsymbol{\lambda}+\mathscr{T} \vdash \boldsymbol{M = N}$; this is also written as $\boldsymbol{M} =_{\mathscr{T}} \boldsymbol{N}$.

(ii) $\mathscr{T} + \boldsymbol{M = N}$ stands for $(\mathscr{T} \cup \{\boldsymbol{M = N}\})^+$.

(iii) $\mathscr{T}\boldsymbol{\eta}$ stands for $(\boldsymbol{\lambda\eta}+\mathscr{T})^+$.

(iv) If $\mathscr{T} = (\mathscr{T}_{\mathbf{0}})^+$, then $\mathscr{T}$ is said to be axiomatized by $\mathscr{T}_{\mathbf{0}}$.

(v) Write $\boldsymbol{x} \in_{\mathscr{T}} \boldsymbol{M}$ if $\forall \boldsymbol{N} =_{\mathscr{T}} \boldsymbol{M},\ \boldsymbol{x} \in \mathrm{FV}(\boldsymbol{N})$.

(vi) $\boldsymbol{M} \in_{\mathscr{T}} \boldsymbol{\mathfrak{X}}$ if $\exists \boldsymbol{N} =_{\mathscr{T}} \boldsymbol{M},\ \boldsymbol{N} \in \boldsymbol{\mathfrak{X}}$.

(vii) $\mathbf{1} \equiv \boldsymbol{\lambda xy.xy}$ (Church's numeral 1)

4.1.5 (Lemma) For a $\boldsymbol{\lambda}$-theory $\mathscr{T}$ one has $\mathscr{T}\boldsymbol{\eta} = \mathscr{T}+$ $(\mathbf{I} = \mathbf{1})$.
Proof:

Note: An important $\boldsymbol{\lambda}$-theory is obtained following the proposal 2.2.14 to identify unsolvable terms.

4.1.6 (Definition) (i) $\mathscr{K} = \{\boldsymbol{M = N | M, N} \in \boldsymbol{\Lambda^0},$ unsolvable$\}$

(ii) $\mathscr{K} = \mathscr{K}_{\mathbf{0}}^{+}$.

4.1.7 (Definition) Let $\mathscr{T}$ be a t $\boldsymbol{\lambda}$-theory.

(i) $\mathscr{T}$ is *r.e.* if after coding $\mathscr{T}$ is a recursively enumerable set of integers.

(ii) $\mathscr{T}$ is *sensible* if $\mathscr{K} \subseteq \mathscr{T}$.

(iii) $\mathscr{T}$ is *semi sensible* (*s.s.*) if $\mathscr{T}$ does not equate a solvable and an unsolvable term.

Note: Both $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda\eta}$ are r.e. and *s.s.* (the latter will be proved in §17.1).

4.1.8 (Lemma) (i) Let $\mathbf{K}^{\infty}$ be a fixed point of $\mathbf{K}$. Then $\mathbf{I} = \#\ \mathbf{K}^{\infty}$.

(ii) (*Jacopini* [1975]) Let $\boldsymbol{\omega_3} \equiv \boldsymbol{\lambda x.xxx}$ and $\boldsymbol{\Omega_3} \equiv \boldsymbol{\omega_3\omega_3}$. Then $\mathbf{I}\ \#_{\boldsymbol{I}}\boldsymbol{\Omega_3}$.

Proof: (i) First note that $\mathbf{K}^{\infty} = \mathbf{K}\mathbf{K}^{\infty} = \mathbf{K}^{\infty}$. Hence

$$\mathbf{I} = \mathbf{K}^{\infty} \vdash \boldsymbol{M} = \mathbf{I}\boldsymbol{M} = \mathbf{K}^{\infty}\boldsymbol{M} = \mathbf{K}^{\infty} = \mathbf{K}^{\infty}\boldsymbol{N} = \mathbf{I}\boldsymbol{N} = \boldsymbol{N}.$$

(ii) Note that $\boldsymbol{\Omega_3} \equiv \boldsymbol{\omega_3\omega_3} = \boldsymbol{\omega_3\omega_3\omega_3} \equiv \boldsymbol{\Omega_3\omega_3}$. Hence

$$\mathbf{I} = \boldsymbol{\Omega_3} \vdash \mathbf{I} = \boldsymbol{\Omega_3} = \boldsymbol{\Omega_3\omega_3} = \mathbf{I}\boldsymbol{\omega_3} = \boldsymbol{\omega_3}.$$

Since $\mathbf{I}$ and $\boldsymbol{\omega_3}$ are different $\boldsymbol{\beta\eta}$-nf's, one has by Bö̈hm's theorem for $\boldsymbol{\lambda}\mathbf{I}$, theorem 10.5.31, that $\mathbf{I}\ \#_{\boldsymbol{I}}\boldsymbol{\omega_3}$. Hence we're done.

4.19 (Corollary) $\mathscr{T}$ sensible $\Rightarrow$ $\mathscr{T}$ semi sensible.
Proof:

Rules

The rule of extensionality (*ext*) and the rule $\boldsymbol{\xi}$ from chapter two. Plus:

4.1.10 (Definition) (i) The $\boldsymbol{\omega}$-*rule* is
$\boldsymbol{\omega} : \forall \boldsymbol{Z} \in \boldsymbol{\Lambda^0 MZ = NZ} \Rightarrow \boldsymbol{M = N}$

(ii) The *term rule* is
$\boldsymbol{tr} : \forall \boldsymbol{Z} \in \boldsymbol{\Lambda^0 MZ = NZ} \Rightarrow \boldsymbol{Mx = Nx}$, for arbitrary x.

4.1.11 (Definition) Let $\mathscr{T}$ be a $\boldsymbol{\lambda}$-theory.

(i) $\mathscr{T}$ is closed under the $\boldsymbol{\omega}$-rule notation $\mathscr{T} \vdash \boldsymbol{\omega}$ if

$$\forall \boldsymbol{Z} \in \boldsymbol{\Lambda^0} \quad \mathscr{T} \vdash \boldsymbol{MZ = NZ} \Rightarrow \mathscr{T} \vdash \boldsymbol{M = N}$$

(ii) Similarly one defines $\mathscr{T} \vdash \boldsymbol{R}$ for the other rules. Note that by definition, for every $\boldsymbol{\lambda}$-theory $\mathscr{T}$ one has $\mathscr{T} \vdash \boldsymbol{\xi}$

(iii) $\mathscr{T}$ is *extensional* if $\mathscr{T} \vdash \boldsymbol{ext}$.

4.1.12 (Lemma) (i) $\mathscr{T} \vdash \boldsymbol{\omega} \Leftrightarrow \mathscr{T} \vdash \boldsymbol{tr}$ and $\mathscr{T} \vdash \boldsymbol{ext}$

(ii) $\mathscr{T} \vdash \boldsymbol{ext} \Leftrightarrow \mathscr{T} \vdash \mathbf{I} = 1 \Leftrightarrow \mathscr{T} = \mathscr{T} = \mathscr{T}\boldsymbol{\eta}$.

Proof:

4.1.13 (Notation) For a $\boldsymbol{\lambda}$-theory $\boldsymbol{mathscrT}$ and a rule $\boldsymbol{R}$ let $\mathscr{T} + \boldsymbol{R}$ of $\mathscr{T}\boldsymbol{R}$ be

$$\{\boldsymbol{M} = \boldsymbol{N} | \boldsymbol{M}, \boldsymbol{N} \in \boldsymbol{\Lambda}^{\mathbf{0}} \text{ and } \boldsymbol{\lambda} + \boldsymbol{R} + \mathscr{T} \vdash \boldsymbol{M} = \boldsymbol{N}\}$$

in the obvious sense.

In general $\mathscr{T}\boldsymbol{R}$ does not need to be a $\boldsymbol{\lambda}$-theory; corollary 15.3.7 shows that $\neg$ Con($\mathscr{T}\boldsymbol{\eta}$) for some $\boldsymbol{\lambda}$-theory $\mathscr{T}$.

4.1.14 (Definition) Let $\boldsymbol{R}^{\mathbf{0}}$ be the rule $\boldsymbol{R}$ restricted to closed terms. E.g. $\boldsymbol{ext}$ is

$$\boldsymbol{Fx} = \boldsymbol{F'x}, \boldsymbol{F}, \boldsymbol{F'} \in \boldsymbol{\Lambda}^{\mathbf{0}} \text{ and } \boldsymbol{x} \notin \text{FV}(\boldsymbol{FF'}) \Rightarrow \boldsymbol{F} = \boldsymbol{F'}$$

4.1.15 (Proposition) Let $\mathscr{T}$ be a $\boldsymbol{\lambda}$-theory. Then

(i) (*Hindley and Longo* [1980]) $\mathscr{T} \vdash \boldsymbol{\omega}^{\mathbf{0}} \Leftrightarrow \mathscr{T} \vdash \boldsymbol{\omega}$

(ii) $\mathscr{T} \vdash \boldsymbol{tr}^{\mathbf{0}} \Leftrightarrow \mathscr{T} \vdash \boldsymbol{tr}$

(iii) $\mathscr{T} \vdash \boldsymbol{ext}^{\mathbf{0}} \not\Rightarrow \mathscr{T} \vdash \boldsymbol{ext}$.

Proof:

Term Models

Term models consist of the set of closed $\boldsymbol{\lambda}$-terms modulo some $\boldsymbol{\lambda}$-theory $\mathscr{T}$ and reflect the properties of such a theory.

4.1.16 (Definition) (i) A *combinatory algebra* is a structure

$$\mathfrak{M} = \langle \boldsymbol{X}, \cdot, \boldsymbol{k}, \boldsymbol{s} \rangle$$

such that the Card($\boldsymbol{X}$) > 1 and $\boldsymbol{kxy} = \boldsymbol{x}, \boldsymbol{sxyz} = \boldsymbol{xz}(\boldsymbol{yz})$ are valid in $\mathfrak{M}$.

Moreover such a structure is *extensional* if in $\mathfrak{M}$

$$(\forall \boldsymbol{x} \;\; \boldsymbol{ax} = \boldsymbol{bx}) \;\rightarrow\; \boldsymbol{a} = \boldsymbol{b}$$

4.1.17 (Definition) Let $\mathscr{T}$ be a $\boldsymbol{\lambda}$-theory.
(i) The (*open*) *term model* of $\mathscr{T}$ is the structure

$$\mathfrak{M}(\mathscr{T}) = \langle \boldsymbol{\Lambda} / =_{\mathscr{T}}, \cdot, [\mathbf{K}]_{\mathscr{T}}, [\mathbf{S}]_{\mathscr{T}} \rangle$$

where for $\boldsymbol{M}, \boldsymbol{N} \in \boldsymbol{\Lambda}$

$$\boldsymbol{M} =_{\mathscr{T}} \boldsymbol{N} \Leftrightarrow \mathscr{T} \vdash \boldsymbol{M} = \boldsymbol{N}$$

$$[\boldsymbol{M}]_{\mathscr{T}} = \{\boldsymbol{N} \in \boldsymbol{\Lambda} | \boldsymbol{M} =_{\mathscr{T}} \boldsymbol{N}\}$$

$$\mathbf{\Lambda}/=_{\mathscr{T}}\ = \{[M]_{\mathscr{T}} | M \in \mathbf{\Lambda}\}$$

$$[M]_{\mathscr{T}} \cdot [N]_{\mathscr{T}} = [MN]_{\mathscr{T}}$$

(ii) Similarly one defines the *closed term model*

$$\mathfrak{M}^0(\mathscr{T}) = \langle \mathbf{\Lambda}^0/=_{\mathscr{T}}, \cdot, [\mathbf{K}]_{\mathscr{T}}, [\mathbf{S}]_{\mathscr{T}} \rangle$$

4.1.18 (Proposition) Let $\mathscr{T}$ be a $\boldsymbol{\lambda}$-theory. Then

(i) $\mathfrak{M}(\mathscr{T})$ and $\mathfrak{M}^0(\mathscr{T})$ are combinatory algebras.

(ii) $\mathscr{T} \vdash \boldsymbol{ext} \Leftrightarrow \mathfrak{M}(\mathscr{T})$ is *extensional.*

(iii) $\mathscr{T} \vdash \boldsymbol{\omega} \Leftrightarrow \mathfrak{M}^0(\mathscr{T})$ is *extensional.*

Proof:

Remark: In general

$\mathscr{T} \vdash \boldsymbol{ext} \not\Rightarrow \mathfrak{M}^0(\mathscr{T})$ is extensional.

This is so because $\boldsymbol{\lambda\eta} \vdash \boldsymbol{ext}$ but not $\boldsymbol{\omega}$.

4.1.19 (Definition)(i) Let $\mathscr{T}$ be a $\boldsymbol{\lambda}$-theory. Then the *canonical map*

$\phi_{\mathscr{T}} : \mathbf{\Lambda} \ \rightarrow\ \mathfrak{M}(\mathscr{T})$ is defined by $\phi_{\mathscr{T}}(\mathfrak{M}) = [\mathfrak{M}]_{\mathscr{T}}$

(ii) If $\mathscr{T}_1, \mathscr{T}_2$ are $\boldsymbol{\lambda}$-theories with $\mathscr{T}_1 \subseteq \mathscr{T}_2$, then the canonical map

$\phi_{\mathscr{T}_1\mathscr{T}_2} : \mathfrak{M}(\mathscr{T}_1) \ \rightarrow\ \mathfrak{M}(\mathscr{T}_2)$ is defined by $\phi_{\mathscr{T}_1\mathscr{T}_2}([M]_{\mathscr{T}_1}) = [M]_{\mathscr{T}_2}$

(iii) Similarly one defines canonical maps

$\phi^0_{\mathscr{T}} : \mathbf{\Lambda}^0 \ \rightarrow\ \mathfrak{M}^0(\mathscr{T})$ and $\phi^0_{\mathscr{T}_1\mathscr{T}_2} : \mathfrak{M}^0(\mathscr{T}_1) \ \rightarrow\ \mathfrak{M}^0(\mathscr{T}_2)$

4.1.20 (Lemma)

Lambda theories are non degenerate congruence relations on $\mathfrak{M}(\boldsymbol{\lambda})$.

Completeness of theories

4.1.22 (Definition) An equational theory $\mathscr{T}$ is called *Hilbert Post* (HP)- *complete* if for every equation $M = N$ in the language of $\mathscr{T}$

$\mathscr{T} \vdash M = N$ or $\mathscr{T} + (M = N)$ is consistent.

The notion applies to particular $\boldsymbol{\lambda}$-theories. HP-complete theories correspond to maximally consistent theories in first order model theory. Although, if $\mathfrak{A}$ is a first order structure then Th($\mathfrak{A}$) is maximally consistent. But if $\mathfrak{M}$ is, say, a combinatory algebra, then

$$\text{Th}(\mathfrak{M}) = \{M = N | \mathfrak{M} \models M = N, M, N \in \Lambda^0\}$$

is not necessarily complete. For example, Th($\mathfrak{M}(\lambda)$) $=\lambda$ and this theory has many proper extensions.

By Zorn's lemma every λ-theory can be extended to a HP-complete one.

5 Models

INTRODUCTORY STUFF!

5.1 Combinatory Algebras

5.1.1 (Definition) (i) $\mathfrak{M} = (X, \cdot)$ is an *applicative structure* if $\cdot$ is a binary operation on X.

(ii) Such a structure is *extensional* if for $a, b \in X$ one has

$$(\forall x \in X \;\; a \cdot x = b \cdot x) \Rightarrow a = b$$

Notation: (i) As in algebra, $a{\cdot}b$ is usually written ab. If $\vec{b} = b_1, \ldots, b_n$, then $a\vec{b} = ab_1, \ldots, b_n = (\ldots((ab_1)b_2)\ldots b_n)$

(ii) If $\mathfrak{M} = (X, \cdot)$ then we write $a \in \mathfrak{M}$ instead of $a \in X$.

5.1.2 (Definition) Let $\mathfrak{M}$ be an applicative structure

(i) The set of *terms over* $\mathfrak{M}$ (denoted by $\mathscr{T}(\mathfrak{M})$) is inductively defined as follows

$v_0, v_1, v_2, \ldots \in \mathscr{T}(\mathfrak{M}),$ (variables)

$a \in \mathfrak{M} \Rightarrow c_a \in \mathscr{T}(\mathfrak{M}),$ (constants)

$A, B \in \mathscr{T}(\mathfrak{M}) \Rightarrow (AB) \in \mathscr{T}(\mathfrak{M})$

Notation: $A, B \ldots$ denote arbitrary terms and $x, y, \ldots$ arbitrary variables in $\mathscr{T}(\mathfrak{M})$

(ii) a *valuation* in $\mathfrak{M}$ is a map ρ: variables $\longrightarrow$ $\mathfrak{M}$. For a valuation ρ in $\mathfrak{M}$ the *interpretation* of $A \in \mathscr{T}(\mathfrak{M})$ in $\mathfrak{M}$ under ρ (denoted by $(A)^{\mathfrak{M}}_{\rho}$ or $(A)_{\rho}$ or $(A)^{\mathfrak{M}}$ if $\mathfrak{M}$ or ρ is clear from the context) is inductively defined as usual:

$$(x)^{\mathfrak{M}}_{\rho} = \rho(x)$$

$$(c_a)^{\mathfrak{M}}_{\rho} = a$$

$$(AB)^{\mathfrak{M}}_{\rho} = (A)^{\mathfrak{M}}_{\rho}(B)^{\mathfrak{M}}_{\rho}$$

(iii) $A = B$ us *true in* $\mathfrak{M}$ *under the valuation* ρ (denoted by $\mathfrak{M}, \rho \models A = B$) if $(A)^{\mathfrak{M}}_{\rho} = (B)^{\mathfrak{M}}_{\rho}$.

(iv) $A = B$ is *true* in $\mathfrak{M}$ (denoted by $\mathfrak{M} \models A = B$) if $\mathfrak{M}, \rho \models A = B$ for all valuations ρ.

(v) The relation $\models$ is also used for first order formulas over $\mathfrak{M}$. The definition is as usual.

Remember that when evaluating a formula, the free variables of that formula are dependent on the interpretation, ρ.

5.1.3 (Definition) (Due to Curry) An applicative structure $\mathfrak{M}$ is a *combinatory complete* if for every $A \in \mathscr{T}(\mathfrak{M})$ and $x_1, \ldots, x_n$ with FV(A)$\subseteq \{x_1, \to, x_N\}$ one has ion $\mathfrak{M}$

$$\exists f \ \forall x_1, \ldots, x_n = A$$

Note that an extensional applicative structure is combinatory complete iff for all $A \in \mathscr{T}(\mathfrak{M})$ one has

$$\exists! f \ \forall \vec{x} f \vec{x} = A(\vec{x})$$

5.1.4 (Notation) Intuitive. . .

5.1.5 (Lemma) Let $\mathfrak{M}$ be an applicative structure and $A, A', B, B' \in \mathscr{T}(\mathfrak{M})$ Then

(i) $(A[x := B])_\rho = (A)_{\rho(x:=(B)_\rho)}$

(ii) $\mathfrak{M} \models A = A' \ \wedge \ B = B' \Rightarrow \mathfrak{M} \models A[x := B] = A'[x := B']$

Proof:

5.1.6 (Definition) Let $\mathfrak{M} = (X, \cdot)$ be an applicative structure and let $\phi : X^n \ \to \ X$ be a map

(i) ϕ is *representable* over $\mathfrak{M}$ if

$$\exists f \in X \ \forall \vec{a} \in \phi(\vec{a})$$

(ii) ϕ is *algebraic* over $\mathfrak{M}$ if there is a term $A \in \mathscr{T}(\mathfrak{M})$ with FV(A) $\subseteq \{x_1, \ldots, x_n\}$ such that

$\forall \vec{a} \ \phi(\vec{a}) = (A)_{\rho(\vec{x}:=\vec{a})}$ Note that this is independent of ρ. Why?

Combinatory completeness says that all algebraic functions are representable. The converse is trivial.

5.1.7 (Definition) A *combinatory algebra* is an applicative structure $\mathfrak{M} = (X, \cdot, k, s)$ with k and s defined as usual.

5.1.8 (Definition) Let $\mathfrak{M}$ be a combinatory algebra.

(i) Extend $\mathscr{T}(\mathfrak{M})$ with new constants K and S denoting k and s respectively. Also $I = SKK$.

(ii)For $A = \mathscr{T}(\mathfrak{M})$ and a variable x, define $\lambda^* x.A \in \mathscr{T}(\mathfrak{M})$ inductively as follows

$\lambda^* x.x = I$

$\lambda^* x.P = KP$, if P does not contain x

$\lambda^* x.PQ = S(\lambda^* x.P)(\lambda^* x.Q)$

(iii) Let $\vec{x} = x_1, \ldots, x_n$. Then $\lambda^* \vec{x}.A = (\lambda^* x_1 \ldots (\lambda^* x_n A) \ldots)$

5.1.9 (Proposition) Intuitive. . .

5.1.10 (Theorem) An applicative structure $\mathfrak{M}$ is combinatory complete iff it can be expanded to a combinatory algebra (by choosing k, s). Hence every combinatory algebra is combinatory complete. *Proof*:

5.1.11 (Remarks) Note that a combinatory algebra $\mathfrak{M} = (X, \cdot, k, s)$ is nontrivial (i.e., Card($\mathfrak{M} > 1$) iff $k \neq s$. Indeed, $k = s$ implies $a = s(ki)(ka)z = k(ki)(ka)z = i$ for all a, so $\mathfrak{M}$ is trivial. We usually just assume that what we are dealing with is nontrivial.

5.1.12 (Definition) (i) Let $\mathfrak{M}_i = (X_i, \cdot_i, k_i, s_i), i = 1, 2,$ be combinatory algebras. Then $\phi : X_1 \rightarrow X_2$ is a *homomorphism* (denoted by $\phi : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$) if ϕ preserves application and k and s, i.e. $\phi(x \cdot_1 y) = \phi(x) \cdot_2 \phi(y), \phi(k_1) = k_2,$ and $\phi(s_1) = s_2$.

(ii)$\mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ if $\phi : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ for some ϕ.

(iii) $\mathfrak{M}$ is *embeddable* in $\mathfrak{M}_2$ (denoted by $\mathfrak{M}_1 \hookrightarrow \mathfrak{M}_2$) if $\phi : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ for some injective ϕ.

$\mathfrak{M}_1$ is a *substructure* of $\mathfrak{M}_2$ (denoted $\mathfrak{M}_1 \subset \mathfrak{M}_2$) with ϕ the identity.

(iv)$\mathfrak{M}_1$ is *isomorphic* to $\mathfrak{M}_2$ (denoted by $\mathfrak{M}_1 \cong \mathfrak{M}_2$) if $\phi : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ for some bijective ϕ.

5.1.13 (Definition) (i) $\mathcal{C}$ is the set of terms of combinatory logic, i.e. applicative terms built up with variables and K, S only.

$$\mathcal{C}^0 = \{P \in \mathcal{C} | \text{ FV}(P) = \emptyset\}$$

(ii) Let $\mathfrak{M}$ be a combinatory algebra. Then

$$\text{Th}(\mathfrak{M}) = \{P = Q | \mathfrak{M} \vDash P = Q, \ P, Q \in \mathcal{C}^0)\}$$

5.1.13 (Proposition) Let $\phi : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$. Then for $P, Q \in \mathscr{T}(\mathfrak{M})$

(i) $\phi(\lceil P \rfloor)^{\mathfrak{M}_1}_{\rho} = \lceil \phi P \rfloor^{\mathfrak{M}_2}_{\phi \circ \rho}$, where $\phi(P)$ results from P by replacing the constants c_a by $c_{\phi(a)}$.

(ii)$\mathfrak{M}_1 \vDash P = Q \Rightarrow \mathfrak{M}_2 \vDash \phi(P) = \phi(Q)$, provided $P < Q \in \mathcal{C}^0$ or ϕ is surjective.

(iii) Th$(\mathfrak{M}_1) \subseteq$ Th$(\mathfrak{M}_2)$

(iv) Th$(\mathfrak{M}_1) =$ Th$(\mathfrak{M}_2)$, provided that ϕ is injective.

Proof:

5.1.15 (Proposition) Combinatory algebras (except the trivial one) are

(i) never commutative

(ii) never associative

(iii) never finite

(iv) never recursive

Proof:

5.2 Lambda Algebras and Lambda Models

Since in a combinatory algebra $\mathfrak{A}$ abstraction can be simulated by $\boldsymbol{k}$ and $\boldsymbol{s}$, it is possible to interpret $\boldsymbol{\lambda}$-terms in $\mathfrak{A}$.

(Notation) Let $\boldsymbol{C}$ be a set of constants. $\boldsymbol{\Lambda(C)}$ is the set $\boldsymbol{\lambda}$-terms possibly containing constants from $\boldsymbol{C}$. The $\boldsymbol{\lambda}$-calculus axioms and rules extend in the obvious way to equations $\boldsymbol{M = N}$ with $\boldsymbol{M, N \in \Lambda(C)}$. For these $\boldsymbol{M, N}$ we still write $\boldsymbol{\lambda \vdash M = N}$. If $\mathfrak{M}$ is an applicative structure, then $\boldsymbol{\Lambda(\mathfrak{M})}$ is $\boldsymbol{\Lambda(\{c_a | a \in \mathfrak{M}\})}$.

5.2.1 (Definition) Let $\mathfrak{M}$ be a combinatory algebra.

(i) Define maps

$$\boldsymbol{CL} : \Lambda(\mathfrak{M}) \;\rightarrow\; \mathscr{T}(\mathfrak{M})$$

$$\boldsymbol{\Lambda} : \mathscr{T}(\mathfrak{M}) \;\rightarrow\; \boldsymbol{\Lambda}(\mathfrak{M})$$

Consider the definition in the middle of page 92.

For $\boldsymbol{M, N \in \Lambda(\mathfrak{M})}$ one defines

$$\lceil M \rfloor_{\rho}^{\mathfrak{M}} = \lceil M_{CL} \rfloor_{\rho}^{\mathfrak{M}}$$

$$\mathfrak{M}, \rho \vDash M = N \Leftrightarrow \lceil M \rfloor_{\rho}^{\mathfrak{M}} = \lceil N \rfloor_{\rho}^{\mathfrak{M}}$$

$$\mathfrak{M} \vDash M = N \Leftrightarrow \mathfrak{M}, \rho \vDash M = N \text{ for all } \rho$$

If $\mathfrak{M}$ is a combinatory algebra and $\boldsymbol{a \in \mathfrak{M}}$, then we write (for example) $\boldsymbol{\lambda x.xa}$ for $\lceil \boldsymbol{\lambda x.c_a} \rfloor^{\mathfrak{A}}$.

Also, not all equations provable in $\boldsymbol{\lambda}$-calculus are true in every combinatory algebra. For example, if $\mathfrak{M}$ is the term model of $\boldsymbol{CL}$, then

$$\mathfrak{M} \nvDash \lambda z.(\lambda x.x)z = \lambda z.z$$

since $\boldsymbol{((\lambda z.(\lambda x.x)z)_C L \equiv S(KI)I}$ and $\boldsymbol{(\lambda z.z)_C L \equiv I}$; but $\boldsymbol{\lambda \vdash \lambda z(\lambda x.x)z = \lambda z.z}$

5.2.2 (Definition) (i) A combinatory algebra $\mathfrak{M}$ is called a $\boldsymbol{\lambda}$-*algebra* if for all $\boldsymbol{A, B \in (T)(\mathfrak{M})}$

$$\lambda \vdash A_{\lambda} = B_{\lambda} \Rightarrow \mathfrak{M} \vDash A = B$$

(ii) A $\boldsymbol{\lambda}$-algebra *homomorphism* is just a combinatory algebra homomorphism.

5.2.3 (Lemma) Let $\mathfrak{M}$ be a combinatory algebra. Then $\mathfrak{M}$ is a $\boldsymbol{\lambda}$-algebra iff for all $\boldsymbol{M}, \boldsymbol{N} \in \boldsymbol{\Lambda}(\mathfrak{M})$

1. $\boldsymbol{\lambda} \vdash \boldsymbol{M} = \boldsymbol{N} \Rightarrow \mathfrak{M} \models \boldsymbol{M} = \boldsymbol{N}$
2. $\mathfrak{M} \models \boldsymbol{K}_{\boldsymbol{\lambda}, \boldsymbol{CL}} = \boldsymbol{K}, \qquad \mathfrak{M} \models \boldsymbol{S}_{\boldsymbol{\lambda}, \boldsymbol{CL}} = \boldsymbol{S}$

Proof:

5.2.4 (Proposition) (i) $\phi : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$, then $\phi \lceil M \rfloor_{\rho}^{\mathfrak{M}_1} = \lceil \phi(\mathfrak{M}) \rfloor_{\phi \circ \rho}^{\mathfrak{M}_2}$, for $\boldsymbol{M} \in \boldsymbol{\Lambda}(\mathfrak{M})$. In particular $\phi \lceil M \rfloor^{\mathfrak{M}_1} = \lceil M \rfloor^{\mathfrak{M}_2}$ for $\boldsymbol{M} \in \boldsymbol{\Lambda}^0$.

(ii) Let $\mathfrak{M}_1 \rightarrow \mathfrak{M}_2$. Then $\mathrm{Th}(\mathfrak{M}_1) \subseteq \mathrm{Th}(\mathfrak{M}_2)$. Thus if $\mathfrak{M}_1$ is a $\boldsymbol{\lambda}$-algebra, so is $\mathfrak{M}_2$

(iii) $\mathfrak{M}_1 \looparrowright \mathfrak{M}_2 \Rightarrow \mathrm{Th}(\mathfrak{M}_1) = \mathrm{Th}(\mathfrak{M}_2)$

Proof: By proposition 5.1.14.

By using Curry's combinatory axioms $\boldsymbol{A}_{\boldsymbol{\beta}}$ one can axiomatize the class of $\boldsymbol{\lambda}$-algebras.

5.2.5 (Theorem)

5.2.6 (Definition) Let $\mathfrak{M}$ be a combinatory algebra. $\mathfrak{M}$ is called *weakly extensional* is for $\boldsymbol{A}, \boldsymbol{B} \in \mathscr{T}(\mathfrak{M})$

$$\mathfrak{M} \models \forall(\boldsymbol{A} = \boldsymbol{B}) \rightarrow \boldsymbol{\lambda x . A} = \boldsymbol{\lambda x . B}$$

The condition of weak extensionality us rather syntactical. Meyer [1980] and Scott [1980] replace it with the following:

5.2.7 (Definition) (i) In a combinatory algebra define $\mathbf{1} = \boldsymbol{S}(\boldsymbol{KI})$

(ii)A $\boldsymbol{\lambda}$-*model* is a $\boldsymbol{\lambda}$-algebra $\mathfrak{M}$ such that the following Meyer-Scott axiom holds in $\mathfrak{M}$

$$\forall \boldsymbol{x}(\boldsymbol{ax} = \boldsymbol{bx}) \rightarrow \mathbf{1a} = \mathbf{1b}$$

5.2.8 (Lemma) Let $\mathfrak{M}$ be a combinatory algebra. Then in $\mathfrak{M}$

(i) $\mathbf{1ab} = \boldsymbol{ab}$

If moreover $\mathfrak{M}$ is a $\boldsymbol{\lambda}$-algebra, then
(ii) $\mathbf{1} = \boldsymbol{\lambda xy . xy}$, hence $\mathbf{1a} = \boldsymbol{\lambda y . ay}$

(iii) $\mathbf{1}(\boldsymbol{\lambda x . A}) = \boldsymbol{\lambda x . A}$, for all $\boldsymbol{A} \in \mathscr{T}(\mathfrak{M})$

(iv) $\mathbf{11} = \mathbf{1}$

Proof:

5.2.9 (Proposition) $\mathfrak{M}$ is a $\boldsymbol{\lambda}$-model iff $\mathfrak{M}$ is a weakly extensional $\boldsymbol{\lambda}$-algebra.
Proof:

5.2.10 (Proposition) Let $\mathfrak{M}$ be a $\boldsymbol{\lambda}$-algebra. Then

$$\mathfrak{M} \text{ is extensional iff } \mathfrak{M} \text{ is weakly extensional and satisfies } \boldsymbol{I} = \boldsymbol{1}$$

Proof:

Terms models, interiors

5.2.11 (Definition) Let $\mathscr{T}$ be a $\boldsymbol{\lambda}$-theory
(i) Define
$\boldsymbol{M} =_{\mathscr{T}} \boldsymbol{N} \Leftrightarrow \mathscr{T} \vdash \boldsymbol{M} = \boldsymbol{N}$ (This is a congruence relation on $\boldsymbol{\Lambda}$)

$[\boldsymbol{M}]_{\mathscr{T}} = \{\boldsymbol{N} \in \boldsymbol{\Lambda} | \boldsymbol{M} =_{\mathscr{T}} \boldsymbol{N}\}$

$\boldsymbol{\Lambda}/\mathscr{T} = \{[\boldsymbol{M}]_{\mathscr{T}} | \boldsymbol{M} \in \boldsymbol{\Lambda}\}$

$[\boldsymbol{M}]_{\mathscr{T}} \cdot [\boldsymbol{N}]_{\mathscr{T}} = [\boldsymbol{MN}]_{\mathscr{T}}$ (This is well-defined)

The *the open model* of $\mathscr{T}$ is $\mathfrak{M}(\mathscr{T}) = \langle \boldsymbol{\Lambda}/\mathscr{T}, \cdot, [\boldsymbol{K}]_{\mathscr{T}}, [\boldsymbol{S}]_{\mathscr{T}} \rangle$

(ii) By restricting everything to closed terms one defines the *closed term model* of $\mathscr{T}$

$$\mathfrak{M}^{\mathbf{0}}(\mathscr{T}) = \langle \boldsymbol{\Lambda}^{\mathbf{0}}/\mathscr{T}, \cdot, [\boldsymbol{K}]^{\mathbf{0}}_{\mathscr{T}}, [\boldsymbol{S}]^{\mathbf{0}}_{\mathscr{T}} \rangle$$

Clearly if $\mathscr{T}$ is consistent (i.e., doesn't prove every equation), then $\mathscr{T} \nvdash \boldsymbol{K} = \boldsymbol{S}$, so $\mathfrak{M}(\mathscr{T})$ and $\mathfrak{M}^{\mathbf{0}}(\mathscr{T})$ are nontrivial. In particular $\mathfrak{M}(\boldsymbol{\lambda})$ and $\mathfrak{M}^{\mathbf{0}}(\boldsymbol{\lambda})$ are nontrivial since they follow from the Church-Rosser Theorem that the theory $\boldsymbol{\lambda}$ is consistent.

5.2.12 (Proposition) Let $\mathscr{T}$ be an extension of $\boldsymbol{\lambda}$-calculus and let $\mathfrak{M}$ be $\mathfrak{M}(\mathscr{T})$ or $\mathfrak{M}^{\mathbf{0}}(\mathscr{T})$.

(i) For $\boldsymbol{M}$ with $\text{FV}(\boldsymbol{M}) = \{\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}\}$ and $\boldsymbol{\rho}$ with $\boldsymbol{\rho}(\boldsymbol{x_i}) = [\boldsymbol{P_i}]^{(\mathbf{0})}_{\mathscr{T}}$ one has

$$\lceil \boldsymbol{M} \rfloor^{\mathfrak{M}}_{\boldsymbol{\rho}} = [\boldsymbol{M}[\vec{\boldsymbol{x}} := \vec{\boldsymbol{P}}]]^{(\mathbf{0})}_{\mathscr{T}}$$

(ii) $\mathfrak{M}(\mathscr{T})$ is a $\boldsymbol{\lambda}$-model.

where $[\vec{\boldsymbol{x}} := \vec{\boldsymbol{P}}]$ denotes simultaneous substitution (refer to exercise 2.4.8)

(ii) $\mathscr{T} \vdash \boldsymbol{M} = \boldsymbol{N} \Rightarrow \mathfrak{M} \vDash \boldsymbol{M} = \boldsymbol{N}$

(iii) $\mathscr{T} \vdash \boldsymbol{M} = \boldsymbol{N} \Leftrightarrow \mathfrak{M} \vDash \boldsymbol{M} = \boldsymbol{N}$, provided that $\mathfrak{M} = \mathfrak{M}(\mathscr{T})$ or that $\boldsymbol{M}, \boldsymbol{N}$ are closed.

Proof:

5.2.13 (Corollary) (i) $\mathfrak{M}^0(\mathscr{T})$ is a $\boldsymbol{\lambda}$-algebra.

Proof:

5.2.14 (Definition) Let $\mathfrak{A}$ be a combinatory algebra.

(i) The *interior* of $\mathfrak{A}$ (denoted by $\mathfrak{A}^0$), is the substructure of $\mathfrak{A}$ generated by $\boldsymbol{k}, \boldsymbol{s}$

(ii) $\mathfrak{A}$ is *hard* $\mathfrak{A}^0 = \mathfrak{A}$

Note that up to isomorphism $\mathfrak{M}^0(\mathscr{T})$ is the interior of $\mathfrak{M}(\mathscr{T})$

5.2.15 (Proposition) Let $\mathfrak{A}$ be a $\boldsymbol{\lambda}$-algebra.

$\mathfrak{M}^0(\mathrm{Th}(\mathfrak{A})) \cong \mathfrak{A}^0$

Let $\mathrm{Th}(\mathfrak{A}.) = \{M = N | M, N \in \mathscr{T}(\mathfrak{A}), M, N$ closed and $\mathfrak{A} \models M = N\}$. Then $\mathfrak{M}^0(\mathrm{Th}(\mathfrak{A}.) \cong \mathfrak{A}$

Proof:

5.2.16 (Proposition) (i) (Barendregt, Koymans [1980]) Every $\boldsymbol{\lambda}$-algebra can be embedded into a $\boldsymbol{\lambda}$-model.

(ii) (Meyer [1982]) Every $\boldsymbol{\lambda}$-algebra is the homomorphic image of a $\boldsymbol{\lambda}$-model.

Proof:

5.2.17 (Theorem) (i) There is a $\boldsymbol{\lambda}$-model that cannot be embedded into an extensional $\boldsymbol{\lambda}$-model.

(ii)There is a combinatory complete applicative structure that cannot be made into a $\boldsymbol{\lambda}$-algebra (by choosing $\boldsymbol{k}, \boldsymbol{s}$)

(iii) There is a $\boldsymbol{\lambda}$-algebra that cannot be made into a $\boldsymbol{\lambda}$-model (by changing $\boldsymbol{k}, \boldsymbol{s}$)

(iv) There is a $\boldsymbol{\lambda}$-model that cannot be made into an extensional one (by collapsing it)

Proof: Refer to Barendregt, Koymans [1980].

5.2.18 (Theorem) Let $M, N \in \boldsymbol{\Lambda}$. Then
(i) $\boldsymbol{\lambda} \vdash M = N \Leftrightarrow M = N$ is true in all $\boldsymbol{\lambda}$-models (or $\boldsymbol{\lambda}$-algebras)

(ii) Let $\mathscr{T}$ be an ectension of the $\boldsymbol{\lambda}$-calculus. Then

$\mathscr{T} \vdash M = N \Leftrightarrow M = N$ is true in all $\boldsymbol{\lambda}$-models satisfying $\mathscr{T}$

(iii) Let $(\boldsymbol{\lambda})_c$ be the classical first order theory axiomatized by the universal closure of

$$\boldsymbol{Kxy = x}$$

$$\boldsymbol{Sxyz = xz(yz)}$$

$$\boldsymbol{K \neq S}$$

$$\forall \boldsymbol{x(ax = bx)} \ \rightarrow \ \boldsymbol{1a = 1b}$$

$$\boldsymbol{A_\beta}$$

Then

$$(\boldsymbol{\lambda})_c \vdash \boldsymbol{M = N} \Leftrightarrow \boldsymbol{\lambda} \vdash \boldsymbol{M = N}$$

Proof:

Models and Rules