Mathematical Logic III
The Joy of Sets
§3.0 - §3.5
Neal Hogan

3 Ordinal and Cardinal Numbers

3.1 Ordinal Numbers

Remember, an ordinal was defined in §1.7 to be a woset $(X, <)$ such that

$$a = \{x \in X \mid x < a\} \qquad \text{for every } a \in X$$

Also, for any two ordinals X, Y either $X \in Y$ or $Y \in X$. As we saw before with ordinals $\in$ is simply $<$ or $\subset$. Every well-ordered set is isomorphic to an ordinal. The first ordinal is 0, the second $1 = \{0\}$, . . . the n'th ordinal is $\{0, 1, 2, \ldots, n-1\}$. The first infintite ordinal is $\omega = \{0,\ 1,\ 2, \ldots,\ n,\ n+1, \ldots\}$. The second infinite ordinal is $\omega + 1 = \{0,\ 1, \ldots,\ n, \ldots,\ \omega\}$. In general, any ordinal $\gamma = \alpha + 1$ (i.e., $\gamma = \alpha \cup \{\alpha\}$) is called a *successor ordinal*. Any ordainal that is not a successor ordinal is called a *limit ordinal*. That is, a limit ordinal is an ordinal with no immediate predicessor (examples: $0,\ \omega,\ \omega^{\omega}, \ldots$).
The following set-theorhetic characterization of ordinals is very usefull. A set X is called *transitive* iff

$$[x \in X \quad \wedge \quad a \in x] \ \rightarrow \ a \in X$$

Lemma 3.1.1 A set X is an ordinal iff it is transitive and totally ordered.
Proof: ($\rightarrow$) Suppose X is an ordinal. By the definition of ordinal X is a woset, $(X,\ \subset)$. Thus, for every $x \in X$, $x = \{a \in X \mid a \subset x\}$. Since $(x \in X \ \rightarrow \ x \subseteq X)$, X is transitive. And since X is an ordinal, we know that it is totally ordered by $\in$. (How? Because X is a woset)
($\leftarrow$) Let X be a transitive set that is totally ordered by $\in$. By the axiom of foundation, X is well-ordered by $\in$. Let $x \in X$. Since X is transitive, $(a \in x \ \rightarrow \ a \in X)$, so $x = \{a \in X \mid a \in x\}$. That is the defintion of an ordinal, thus X is an ordinal.

Lemma 3.1.2 If α is an ordinal, then $\alpha \cup \{\alpha\}$ is an ordinal.
Proof: If α is transitive and totally ordered by $\in$ (it is, because it is an ordinal), so too is $\alpha \cup \{\alpha\}$. Thus, by Lemma 3.1.1, $\alpha \cup \{\alpha\}$ is an ordinal.

Lemma 3.1.3 If A is a set of ordinals, then $\bigcup A$ is an ordinal.
Proof: Let $x \in a \in \bigcup A$. For some $b \in A$, $a \in b$. Since b is an ordinal, $x \in a \in b$ implies $x \in b$. Hence $x \in \bigcup A$ Thus, $\bigcup A$ is transitive.
Now, let $x,\ y \in \bigcup A$. Pick $a,\ b \in A$ with $x \in a$ and $y \in b$. Either $a \subseteq b$ or $b \subseteq a$. Assume $a \subseteq b$. Then $x,\ y \in b$. So, either $x \in y$ or $y \in x$ (or $x = y$) (i.e., *connectedness*). Thus, $\bigcup A$ is totally ordered. So, $\bigcup A$ is an ordinal.

Now that we've seen that *ZF* garauntees us the existence of all the finite ordinals, we must now show that the axioms will get us ω. The existence of ω is garaunteed by the axiom of infinity. However, the actual construction on the set ω is as follows:
Proof: Let $\phi(v_n,\ v_m)$ be a formula of LAST s.t. v_m is the successor of v_n. By the axiom of replacement v_m is a set. By the axiom of subset selection we can define ω to be the infinite set of successors just mentioned.

Now that we have ω, the ordinals $\omega + 1,\ \omega + 2,\ \ldots$ follow using Lemma 3.1.2.

The next limit ordinal is $\omega + \omega$ (i.e., $\{0,\ 1,\ \ldots,\ \omega,\ \omega + 1,\ \ldots\}$ How do we prove that $\omega + \omega$ exists?
Let the 'function'

$$f \ : \ \omega \ \rightarrow \ V$$

be defined by

$$f(n) = \omega + n$$

By the axiom of replacement, the collection

$$E = \{f(n) \mid n \in \omega\}$$

is a set. Let

$$A = \bigcup E$$

By the axiom of union, A is a set. By Lemma 3.1.3, A is an ordinal. Clearly, A is the ordinal $\omega + \omega$.
How is this clear? Well, ω just *is* the set of all the ordinals $\mu < \omega$. So, $f(n) = \omega + n$, for all $n < \omega$. Thus, $f(\omega) = \omega + \omega$.
Thus we are able to prove the ordinals $\omega + \omega + 1,\ \omega + \omega + \omega + 2,\ \ldots,\ \omega + \omega + \omega,\ \ldots$.

3.2 Addition of Ordinals

Given ordinals, α and β, we define the *ordinal sum* $\alpha + \beta$. Let

$$A = (\alpha \times \{0\}) \cup (\beta \times \{1\})$$

(think of this as α being a set of blue ordinals (those paired with '0') and β being a set of red ordinals (those paired with '1')),

and we define a well-ordering of A by

$$(\nu,\ i) <_A (\tau,\ j) \leftrightarrow (i < j) \vee (i = j \wedge \nu < \tau).$$

How is this a well-ordering? Well, it's a reverse lexicographic ordering.
We then set

$$\alpha + \beta = \mathrm{Ord}(A, <_A).$$

Generally $\alpha + n$ is the n'th ordinal beyond (i.e., the n successor of) α.

Lemma 3.2.1 Ordinal addition is associative; that is, for all $\alpha,\ \beta,\ \gamma$,

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

Proof: First, consider $\alpha + (\beta + \gamma)$.

$$(\beta + \gamma) = (\beta \times \{0\}) \cup (\gamma \times \{1\}).$$

Now,

$$\alpha + (\beta + \gamma) = (\alpha \times \{0\}) \cup ([(\beta \times \{0\}) \cup (\gamma \times \{1\})] \times \{1\}).$$

Thus, we have the following set:

$$A = \{<\alpha,\ 0>,\ <<\beta,\ 0>,\ 1>,\ <<\gamma,\ 1>,\ 1>\}$$

Notice that this set is well-ordered according to the above definition. Also, A is *order isomorphic*; i.e. there is an order preserving, bijective fuction from A to

$$\{0,\ 1,\ 2,\ 3\}$$

Thus, since A is transative and totally ordered it is an ordinal, by theorem 3.1.1. The same reasoning can be used to evaluate $(\alpha + \beta) + \gamma$. By theorem 1.7.10, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

Example: Suppose $\alpha = 2$, $\beta = 3$, and $\gamma = 4$. The claim is that

$$2 + (3 + 4) = (2 + 3) + 4$$

Remember, 2, 3, and 4 are ordinals, so

$$2 = \{0, 1\},\ 3 = \{0, 1, 2\},\ \text{and}\ 4 = \{0, 1, 2, 3\}$$

Thus,

$$(3+4) = \{\langle 0,0\rangle, \langle 1,0\rangle, \langle 2,0\rangle\} \cup \{\langle 0,1\rangle, \langle 1,1\rangle, \langle 2,1\rangle, \langle 3,1\rangle\}$$

which is

$$\{\langle 0,0\rangle, \langle 1,0\rangle, \langle 2,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle, \langle 2,1\rangle, \langle 3,1\rangle\}.$$

According to the ordering above $3 < 4$.
Now, consider $2 + (3 + 4)$. We get

$$\{\langle 0,0\rangle, \langle 1,0\rangle, \langle\langle 0,0,\rangle,1\rangle, \langle\langle 1,0\rangle,1\rangle, \langle\langle 2,0\rangle,1\rangle, \langle\langle 0,1\rangle,1\rangle, \langle\langle 1,1\rangle,1\rangle, \langle\langle 2,1\rangle,1\rangle, \langle\langle 3,1\rangle,1\rangle\}$$

This set is well-ordered by the definition and it is order isomorphic to $\{0,1,2,3,4,5,6,7,8\}$. Use similar reasoning to evaluate $(2 + 3) + 4$ and you will notice that it too is order isomorphic to $\{0,1,2,3,4,5,6,7,8\}$. Thus, $2 + (3 + 4) = (2 + 3) + 4$.

Ordinal addition is **not** commutative! Consider the following,

$$2 + \omega = \{a,\ b,\ 1,\ 2,\ \ldots\} = \omega$$

Notice that $\{a,\ b,\ 1,\ 2,\ \ldots\}$ is isomorphic to $\{1,\ 2,\ 3,\ 4,\ \ldots\}$ (because $2+\omega = \{\langle a,0\rangle, \langle b,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 2,0\rangle, \ldots\}$) but

$$\omega + 2 = \{1,\ 2,\ \ldots,\ a,\ b\} > \omega$$

(Notice that $(\omega + 2)$ has a last element, but $(2 + \omega)$ doesn't!)

In general,

$$n + \omega = \omega$$

whereas

$$\omega < \omega + 1 < \omega + 2 < \omega + 3 < \ldots$$

Why? Well, because ω is a limit ordinal. It has infinitely many predecessors, but **no** immediate predecessor. Thus, $n + \omega = \omega$. Also, ω is the "beginning" of an infinite sequence of ordinals. Thus, adding something to ω is always greater than ω.

3.3 Multiplication of Ordinals

Let λ be an ordinal, and let $\langle \alpha_\eta | \eta < \lambda\rangle$ be a λ-sequence of ordinals. The *ordinal sum*

$$\textstyle\sum_{\eta<\lambda} \alpha_\eta$$

is defined as follows. Set

$$A = \textstyle\bigcup_{\xi<\lambda} (\alpha_\xi\ \times\ \{\xi\}).$$

That is,

$$A = \{\langle 0,0\rangle, \langle 1,0\rangle, \ldots, \langle \alpha,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle, \ldots, \langle \alpha,1\rangle, \langle \alpha,2\rangle, \langle \alpha,3\rangle, \ldots, \langle \alpha,\lambda-1\rangle\}$$

Define a well ordering of A by

$$(\nu,\ \xi) <_A (\nu',\ \xi') \ \leftrightarrow\ (\xi < \xi')\ \vee\ (\xi = \xi'\ \wedge\ \nu < \nu').$$

Again, this is reverse lexicographic ordering.
Let

$$\textstyle\sum_{\xi<\lambda} \alpha_\xi = \mathrm{Ord}(A,\ <_A)$$

Intuitively, the picture looks something like the following, for ordnals $\xi < \lambda$

$$\begin{aligned}
\textstyle\sum_{\xi<2}\alpha_\xi &= \alpha_0+\alpha_1,\\
\textstyle\sum_{\xi<3}\alpha_\xi &= \alpha_0+\alpha_1+\alpha_2,\\
&\vdots\\
\textstyle\sum_{\xi<n}\alpha_\xi &= \alpha_0+\alpha_1+\alpha_2+\ldots+\alpha_{n-1}.\\
&\vdots\\
\textstyle\sum_{\xi<\lambda}\alpha_\xi &= \alpha_0+\alpha_1+\ldots+\alpha_{\lambda-1}
\end{aligned}$$

Notice

$$\textstyle\sum_{n<\omega} n = \sum_{n<\omega} 1 = \omega$$

Why? Well, ω is a limit ordinal. It *is* the sum of all ordinals less than itself.
Finally, we can define *ordinal multiplication* as iterated addition.

$$\alpha\cdot\beta = \textstyle\sum_{\xi<\beta}\alpha$$

In other words,

$$\alpha\cdot\beta = \underbrace{\alpha+\alpha+\ldots+\alpha}_{\beta\ times}$$

Ordinal multiplication is **not** commutative. Consider the following,

$$2\cdot\omega = \underbrace{2+2+\ldots+2}_{\omega\ times} = \omega \quad \text{but} \quad \omega\cdot 2 = \omega+\omega > \omega$$

Why? Well, again it is because ω is a limit ordinal. The equation on the left-hand side of the 'but' says to add the ordinal 2, ω times. Whereas the right-hand side says to add ω twice.

Lemma 3.3.1 (Ordinal multiplication distributive law) For any α, β, and γ,

$$\alpha\cdot(\beta+\gamma) = \alpha\cdot\beta+\alpha\cdot\gamma$$

Proof: First, consider $\alpha\cdot(\beta+\gamma)$. $(\beta+\gamma)$ is evaluated as in 3.2, to be isomorphic to some ordinal, say, μ. Then,

$$\textstyle\sum_{\xi<\mu}\alpha$$

Thus, we have

$$B = \{<\alpha,0>,<\alpha,1>,\ldots,<\alpha,\mu-1>\}$$

Notice that B is well-ordered according to its definition. Thus, B is isomorphic to an ordinal.

Now consider $(\alpha\cdot\beta+\alpha\cdot\gamma)$. We have the follow by the definition for ordinal multiplication

$$\textstyle\sum_{\rho<\beta}\alpha \ + \ \sum_{\tau<\gamma}\alpha$$

Thus we have

$$\{\langle\langle\alpha,0\rangle,0\rangle,\langle\langle\alpha,1\rangle,0\rangle,\ldots,\langle\langle\alpha,\beta-1\rangle,0\rangle\}\cup\{\langle\langle\alpha,0\rangle,1\rangle,\langle\langle\alpha,1\rangle,1\rangle,\ldots,\langle\langle\alpha,\gamma-1\rangle,1\rangle\}$$

which is

$$C = \{\langle\langle\alpha,0\rangle,0\rangle,\langle\langle\alpha,1\rangle,0\rangle,\ldots,\langle\langle\alpha,\beta-1\rangle,0\rangle,\langle\langle\alpha,0\rangle,1\rangle,\langle\langle\alpha,1\rangle,1\rangle,\ldots,\langle\langle\alpha,\gamma-1\rangle,1\rangle\}$$

Notice C is well-oredered according to the definition. Thus, C is isomorphic to an ordinal.
Now, are B and C isomorphic to the same ordinal? Well, $(\beta+\gamma)$ is an ordinal, μ. In $\alpha\cdot(\beta+\gamma)$, α was added to itself $(\beta+\gamma)$ times (i.e., μ times). In $(\alpha\cdot\beta+\alpha\cdot\gamma)$, α is first added to itself β times and then unioned with α added to itself γ times ...
(Note: This proof probably goes a lot smoother using a similar strategy used to prove the associativity of ordinal multiplication given to us by Schlipf (provided here in the nest lemma). The above proof was my attempt at a proof using Devlin's apparatus. I'm confident that I am on the right track, however I'm equally confident that I'm taking the long way!)

Example: Let $\alpha = 2$, $\beta = 3$ and $\gamma = 4$. Consider $2\cdot(3+4)$. $(3+4) = \{0,1,2,3,4,5,6\} = 7$. So,

$$\sum_{\mu<7} 2$$

Which is

$$D = \{\langle 0,0\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \dots, \langle 0,7\rangle, \langle 1,7\rangle\}$$

Notice D is well-ordered according to the definition. Thus there is a function, f, that maps D onto the ordinal

$$14 = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13\}$$

Now consider, $(2 \cdot 3 + 2 \cdot 4)$. We have the following

$$\sum_{\xi<3} 2 \ + \ \sum_{\nu<4} 2$$

Which is

$$E = \{\langle\langle 0,0\rangle,0\rangle, \langle\langle 1,0\rangle,0\rangle, \langle\langle 0,1\rangle,0\rangle, \langle\langle 1,1\rangle,0\rangle, \langle\langle 0,2\rangle,0\rangle, \langle\langle 1,2\rangle,0\rangle, \langle\langle 0,0\rangle,1\rangle, \langle\langle 1,0\rangle,1\rangle, \dots, \langle\langle 0,3\rangle,1\rangle, \langle\langle 1,3\rangle,1\rangle\}$$

Notice E is well-ordered and therefore there is a function mapping E to an ordinal. and that ordinal is

$$14 = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13\}$$

This is the same ordinal that $2 \cdot (3+4)$ is isomorphic to! Thus, $2 \cdot (3+4) = (2 \cdot 3 + 2 \cdot 4)$.

Note: The other distributivity law is false. For example,

$$(1+1) \cdot \omega = 2 \cdot \omega = \omega \quad \text{but} \quad 1 \cdot \omega + 1 \cdot \omega = \omega + \omega > \omega$$

Lemma 3.3.2 (Ordinal multiplication associative law) For any α, β, and γ,

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

Proof: $(\alpha \cdot \beta)$ is order isomorphic to

$$\{\langle \mu, \nu \rangle \mid \mu < \alpha, \ \nu < \beta\}$$

(Note: the set is ordered *co-lexicographically*; i.e. a set of ordered pairs ordered lexicographically from right to left)

So, $(\alpha \cdot \beta) \cdot \gamma$ is order isomorphic to

$$\{\langle\langle \mu, \nu \rangle, \eta\rangle \mid \mu < \alpha, \nu < \beta, \eta < \gamma\}$$

(Again, ordered co-lexicographically)
Likewise for $\alpha \cdot (\beta \cdot \gamma)$. It is order isomorphic to

$$\{\langle \mu, \langle \nu, \eta\rangle\rangle \ \mid \mu < \alpha, \nu < \beta, \eta < \gamma\} \qquad \text{(odered co-lexically)}$$

Now, the claim is that there is an order preserving, 1-1 and onto function,

$$f : \{\langle\langle \mu, \nu\rangle, \eta\rangle\} \ \rightarrow \ \{\langle \mu, \langle \nu, \eta\rangle\rangle\}$$

First, consider when

$$\{\langle\langle \mu_1, \nu_1\rangle, \eta_1\rangle\} < \{\langle\langle \mu_2, \nu_2, \rangle \eta_2\rangle\}$$

This is the case when

$$\text{either } \eta_1 < \eta_2 \text{ or } \eta_1 = \eta_2 \text{ and } \langle \mu_1, \nu_1\rangle < \langle \mu_2, \nu_2\rangle$$

Now,

$$\langle \mu_1, \nu_1\rangle < \langle \mu_2, \nu_2\rangle$$

when

$$\text{either } \nu_1 < \nu_2 \text{ or } \nu_1 = \nu_2, \text{ and } \mu_1 < \mu_2.$$

So, $\{\langle\langle \mu_1, \nu_1\rangle, \eta_1\rangle\} < \{\langle\langle \mu_2, \nu_2, \rangle \eta_2\rangle\}$ when

$$\text{either } \eta_1 < \eta_2, \text{ or } \eta_1 = \eta_2 \text{ and } \nu_1 < \nu_2 \text{ or, } \eta_1 = \eta_2 \text{ and } \nu_1 = \nu_2 \text{ and } \mu_1 < \mu_2$$

Second, consider when

$$\{\langle\mu_1,\langle\nu_1,\eta_1\rangle\rangle\} < \{\langle\mu_2,\langle\nu_2,\eta_2\rangle\rangle\}$$

Well, it is

$$\text{either } \langle\nu_1,\eta_1\rangle < \langle\nu_2,\eta_2\rangle \text{ or } \langle\nu_1,\eta_1\rangle = \langle\nu_2,\eta_2\rangle \text{ and } \mu_1 < \mu_2.$$

Now, $\langle\nu_1,\eta_1\rangle < \langle\nu_2,\eta_2\rangle$ when

$$\text{either } \eta_1 < \eta_2 \text{ or } \eta_1 = \eta_2 \text{ and } \nu_1 < \nu_2.$$

So, $\{\langle\mu_1,\langle\nu_1,\eta_1\rangle\rangle\} < \{\langle\mu_2,\langle\nu_2,\eta_2\rangle\rangle\}$ when

$$\text{either } \eta_1 < \eta_2 \text{ or } \eta_1 = \eta_2 \text{ and } \nu_1 < \nu_2 \text{ or } \nu_1 = \nu_2 \text{ and } \eta_1 = \eta_2 \text{ and } \mu_1 < \mu_2$$

Notice that both sets are ordered under the same conditions!!
So, we have the following:

$$\mathbf{X_1} = [((\alpha \times \beta) \times \gamma), <_{\mathbf{co-lex}}]$$
$$\mathbf{X_2} = [(\alpha \times (\beta \times \gamma)), <_{\mathbf{co-lex}}]$$

and

$$f : \mathbf{X_1} \rightarrow \mathbf{X_2}$$

By

$$f(\langle\langle\mu,\nu\rangle,\eta\rangle) = \langle\mu,\langle\nu,\eta\rangle\rangle$$

Notice that f is 1-1 and onto (If a function has an inverse function, then it is 1-1 and onto. f has an inverse function). So, pick any two pairs from each set and show that the function is order preserving. That is, show that $\langle\langle\mu_1,\nu_1\rangle,\eta_1\rangle < \langle\langle\mu_2,\nu_2\rangle,\eta_2\rangle$ iff $\langle\mu_1,\langle\nu_1,\eta_1\rangle\rangle < \langle\mu_2,\langle\nu_2,\eta_2\rangle\rangle$ We just showed that both sets are well-ordered by the same conditions, so we could easily show f is order preserving. Thus, ordinal multiplication is associative.

3.4 Sequences of Ordinals

Let λ be a limit ordinal, and let $\langle\alpha_\xi \mid \xi < \lambda\rangle$ be a λ-sequence of ordinals. (Note: the sequence is infinite.) We write

$$\alpha = \lim_{\xi<\lambda} \alpha_\xi \text{ iff } (\forall\beta < \alpha)(\exists\xi < \lambda)(\forall\zeta)(\xi < \zeta < \lambda \rightarrow \beta < \alpha_\zeta \leq \alpha)$$

Basically, this is just like a limit in calculus. It is an approximation. So, the sentence says that you pick how good of an approximation you want to make, $\beta < \lambda$, and there is a position in the sequence, ζ such that if you pick any α_ζ in the sequence you will have a *good enough* approximation of the limit of the sequence, α.

Lemma 3.4.1 Let λ be a limit ordinal, and let $\langle\alpha_\xi \mid \xi < \lambda\rangle$ be an increasing sequence of ordinals (notice that the sequence is infinite). Then this sequence has a unique limit

$$\lim_{\xi<\lambda} \alpha_\xi = \bigcup_{\xi<\lambda} \alpha_\xi$$

Proof: Suppose $\beta < \bigcup_{\xi<\lambda} \alpha_\xi$. Then we know that $\beta \in \bigcup_{\xi<\lambda} \alpha_\xi$, by the definition of union. So, we know that β is an element of one of the α_ξ's. Thus, $\beta < \alpha_x i \leq \alpha$, by the property of ordinals being ordered by $<$. Notice that this is exactly the property of limits defined above.

Lemma 3.4.2 Let λ and μ be limit ordinals, and let $f : \mu \rightarrow \lambda$ be an order-preserving function such that $\lim_{\xi<\mu} f(\xi) = \lambda$. Let $\langle\alpha_\xi \mid \xi < \lambda\rangle$ be an increasing sequence. Then

$$\lim_{\xi<\lambda} \alpha_\xi = \lim_{\xi<\mu} \alpha_{f(\xi)}$$

Intuitive Proof: Basically, the lemma says that if you have two infinte, increasing sequences, one of which, in this case the sequence indexed by μ, is the product of a function that generates a *subsequence* of the other (and it is *co-final*) then both sequences have the same limit. Consider the following example.
Let

$$\langle\alpha_\xi \mid \xi < \lambda\rangle = \langle 0, 1, 2, 3, 4, \ldots\rangle$$

and let $f : \mu \;\rightarrow\; \lambda$ pick out all of the even elements of the above sequence, such that

$$\langle f(\alpha_\xi) \mid \xi < \mu \rangle = \langle 0, 2, 4, 6, \dots \rangle$$

It should be obvious that the two sequences have the same limit.

Lemma 3.4.3 Let λ be a limit ordinal, and let $\langle \alpha_\xi \mid \xi < \lambda \rangle$, $\langle \beta_\zeta \mid \zeta < \lambda \rangle$ be increasing sequences such that

(a) $(\forall \xi < \lambda)(\exists \zeta < \lambda)(\beta_\zeta > \alpha_\xi)$
(b) $(\forall \zeta < \lambda)(\exists \xi < \lambda)(\alpha_\xi > \beta_\zeta)$
Then

$$\lim_{\xi<\lambda} \alpha_\xi = \lim_{\zeta<\lambda} \beta_\zeta$$

Intuitive Proof: Basically, what this lemma is saying is that if you take two infinitely increasing, co-final sequences that intersects infinitely many times (that is what (a) and (b) are saying), then the two sequences have the same limit. Consider the sequence consisting of all the points of intersection. This too is an infinitely increasing, co-final sequence. Actually, it is a subsequence that the two sequences in question share. Thus, by lemma 3.4.2, $\lim_{\xi<\lambda} \alpha_\xi = \lim_{\zeta<\lambda} \beta_\zeta$.

Lemma 3.4.4 Let λ be a limit ordinal and let $\langle \alpha_\xi \mid \xi < \lambda \rangle$ be a λ-sequence of ordinals (i.e., an infinite sequence). For each $\mu < \lambda$, let

$$\sigma_\mu = \sum_{\xi<\mu} \alpha_\xi$$

Then

$$\sum_{\xi<\lambda} \alpha_\xi = \lim_{\mu<\lambda} \sigma_\mu$$

Intuitive Proof: Basically, if you take an infinite sequence of ordinals and consider the sequence generated by taking an infinite partial sums of a series with the same limit, then the two sequences have the same limit. Consider the following example (for I feel that what I've just stated is unclear.
Look at the following sequence

$$\langle 3, 3.1, 3.14, 3.141, 3.1415, \dots \rangle$$

Clearly, the limit of this sequence is π.
Now, consider the following series

$$[3 + .1 + .04 + .001 + .0005 + \dots]$$

The sequence of partial sums looks like

$$\langle 3, 3 + .1, 3 + .1 + .04, 3 + .1 + .04 + .001, 3 + .1 + .04 + .001 + .0005, \dots \rangle$$

which is

$$\langle 3, 3.1, 3.14, 3.141, 3.1415, \dots \rangle$$

It should be clear the the limit of both sequences is π.

Two Definitions!
(1) Let $f : \lambda \;\rightarrow\; \lambda$, and let $\alpha \in \lambda$ be a limit ordinal. We say f is *continuous* at α iff

$$f(\alpha) = \lim_{\xi<\alpha} f(\xi)$$

That is, f is continuous if the limit exists at every limit ordinal in the sequence in question. The identity function is an example of a continuous function.

(2) A function $f : \lambda \;\rightarrow\; \lambda$ is said to be a *normal function* iff it is both order-preserving and continuous at every limit ordinal in λ.
(Note: Schlipf remarked that he has never heard of this term before so be aware that it may not be common in the literature.)

Exercise 3.4.1 This exercise is left for you to figure out for I do not know topology! Sorry!

Lemma 3.4.5 Let $f : \mu \;\rightarrow\; \mu$ be a normal function, and let $\lambda \in \mu$ be a limit ordinal. If $\langle \alpha_\xi \mid \xi < \lambda \rangle$ is an increasing sequence of ordinals in μ and $\lim_{\xi<\lambda} \alpha_\xi < \mu$, then

$$f(\lim_{\xi<\lambda}\alpha_\xi) = \lim_{\xi<\lambda} f(\alpha_\xi)$$

Proof:

$$f(\lim_{\xi<\lambda}\alpha_\xi) = f(\bigcup_{\xi<\lambda}\alpha_\xi) \text{ (by substituting the definition for limit)}$$
$$f(\bigcup_{\xi<\lambda}\alpha_\xi) = \lim_{\gamma<\alpha} f(\gamma) \text{ (this follows by continuity if we let } \alpha = \bigcup_{\xi<\lambda}\alpha_\xi)$$

Thus, according to lemma 3.4.2

$$\lim_{\gamma<\alpha} f(\gamma) = \lim_{\xi<\lambda} f(\alpha_\xi)$$

as required.

Lemma 3.4.6 Let $f : \lambda \to \lambda$ be order-preserving. Then $f(\alpha) \geq \alpha$ for all $\alpha \in \lambda$.
Proof: This is just a reformulation of Theorem 1.7.2.

Let $f : \lambda \to \lambda$. We say $\alpha \in \lambda$ is a *fixed-point* of f iff $f(\alpha) = \alpha$.

Lemma 3.4.7 [Fixed-Point Theorem] Let $f : \mathrm{On} \to \mathrm{On}$ be a normal function (in a class sense because we are talking about the collection of all ordinals). For every α there is a fixed-point γ of f such that $\gamma \geq \alpha$.
Proof: Let α be given. If $f(\alpha) = \alpha$, then we're done. So assume otherwise. Then, by lemma 3.4.6, $f(\alpha) > \alpha$. By recursion, we can define a function $g : \omega \to \mathrm{On}$ so that

$$g(0) = \alpha$$
$$g(n+1) = f(g(n))$$

We don't know what α is but we know that g is order-preserving. Consider $g(1)$. We don't know what this maps to but we know $g(1) > g(0)$ because $g(1) = f(g(0)$ and f is order-preserving. Now, by lemma 3.4.1, since we are dealing with an infinitely increasing sequence there exists a limit, namely $\gamma = \lim_{n<\omega} g(n)$. What we have left to show is that $f(\gamma) = \gamma$. Since f is normal we have by lemma 3.4.5

$$f(\gamma) = f(\lim_{n<\omega} g(n)) \text{ [\textit{by substitution}]} = \lim_{n<\omega} f(g(n)) \text{ [\textit{by lemma 3.4.5}]} = \lim_{n<\omega} g(n+1) \text{ [\textit{by substitution}]} = \gamma \text{ [\textit{since it is the limit of an } \omega\text{\textit{-sequence}}]}$$

3.5 Ordinal Exponentiation

Let $\alpha \in \mathrm{On}$. By recursion, we define α^β for some ordinal β

$\alpha^0 = 1$
$\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ (i.e., $\alpha^{\beta-1} \cdot \alpha \cdot \alpha = \alpha^{\beta-2} \cdot \alpha \cdot \alpha \cdot \alpha = \ldots = \alpha^{\beta-\beta} \cdot \underbrace{\alpha \cdot \alpha \cdot \ldots \cdot \alpha}_{(\beta-1)\ times}$
$\alpha^\beta = \lim_{\gamma<\beta}\alpha^\gamma$, if β is a limit ordinal.

Note: I have yet to prove lemmas 3.5.1 and 3.5.2 as well as complete the final exercises. Also, Schlipf pointed out that, as far as he can tell, ordinal exponentiation is not that widely discussed. Cardinal exponentiation plays a much bigger role in the field. Thus, I feel rather justified in not providing all of the details of this section. An intuitive idea of ordinal exponentiation should be sufficient at this point.

One final question, one which Schlipf suggested we ponder, is whether the following ordinal is countable

$$\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$$

GOOD LUCK!!