A Mathematical Introduction to Logic
by Herbert B. Enderton
Annotated Notes and Ramblings of a Tired Student

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These Notes are ROUGH and incomplete!
Once I have some free time they will be corrected and completed!

2 First-Order Logic

2.2 Truth And Models

> In sentential logic, we had truth assignments to tell us which sentence symbols were to be interpreted as being true and whcih as false. In first-order logic, the analogous role is played by *structures*, which can be thought of as providing the dictionary for translations from the formal language into English. Structures are sometimes called *interpretations*, but we prefer to reserve that word for another concept encountered later.

For a first-order language, a structure will consist of two things:

1. What collection of things the universal quantifier symbol ($\forall$) refers to (i.e. the universe our structure is describing), and

2. What the other parameters (the predicate and function symbols) denote

We may represent a structure $\mathfrak{A}$ as follows:

$$\mathfrak{A} = (U;\ P,\ f,\ c) \tag{1}$$

Where each symbol represents the following:

- $U^{\mathfrak{A}}$ represents the universe (sometimes abbreviated $|\mathfrak{A}|$) and is the set of all objects that $\forall$ quantifies over
- $P^{\mathfrak{A}}$ represents some n-ary relation on U
- $f^{\mathfrak{A}}$ represents some n-ary function on U
- $c^{\mathfrak{A}}$ represents some constant symbol

If this doesn't make immediate sense, the following example should clear things up.

Example. Let's assume we want to work in a system of basic number theory. Assume our reduct of number theory contains the operations S (successor function) and $+$ (addition), the binary relation $\leq$, a constant symbol 0, and that our universe is $\mathbb{N}$ (the set of all natural numbers). Our structure $\mathfrak{A}$ would then be represented as follows:

$$\mathfrak{A} = (\mathbb{N};\ \leq,\ S,\ +,\ 0) \tag{2}$$

Now, certain statements may be true in one model and not another. To express this, we introduce the following notation, that may be read as "σ is true in $\mathfrak{A}$".

$$\models_{\mathfrak{A}} \sigma \tag{3}$$

Theorem. *Assume that s_1 and s_2 are functions from V (the set of all variables) into $|\mathfrak{A}|$ which agree at all variables (if any) that occur in the wff ϕ. Then*

$$\models_{\mathfrak{A}} \phi[s_1] \quad \textit{iff} \quad \models_{\mathfrak{A}} \phi[s_2] \tag{4}$$

Corollary. For a sentence σ, either

1. $\mathfrak{A}$ satisfies σ with every function s from V into $|\mathfrak{A}|$

2. $\mathfrak{A}$ does not satisfy σ with any such function

If (1) holds, then it is said that σ is *true* in $\mathfrak{A}$ (writtten $\models_{\mathfrak{A}} \sigma$) or that $\mathfrak{A}$ is a *model* of σ. If (2) holds, then σ is *false* in $\mathfrak{A}$. Notice that they cannot both hold since $|\mathfrak{A}|$ cannot be empty.
Further, it is said that $\mathfrak{A}$ is a model of a set Σ of sentences iff it is a model of every member of Σ.

Example. Let $\mathfrak{R}$ be the real field (number theory with real numbers), with $\mathfrak{R} = (\mathbb{R};\ 0,\ 1,\ +,\ \times)$, and let $\mathfrak{Q}$ be the rational field (number theory with rational numbers only), with $\mathfrak{Q} = (\mathbb{Q};\ 0,\ 1,\ +,\ \times)$. Is there a sentence σ true in one and false in the other?
Yes. Let $\sigma = \exists x(x \cdot x = 1 + 1)$. The only possible value for x in σ is $\sqrt{2}$, wich is not a rational number but is a real number. Thus, $\models_{\mathfrak{R}} \sigma$ but $\not\models_{\mathfrak{Q}} \sigma$.

Enderton now begins formally developing a notion of logical implication. In Endertion, logical implication is defined as follows:

Definition (Logical Implication & Equivalence). Let Γ be a set of wffs, and ϕ a wff. Then Γ *logically implies* ϕ, written $\Gamma \models \phi$ iff for every structure $\mathfrak{A}$ for the language and every function $s : V \to |\mathfrak{A}|$ such that $\mathfrak{A}$ satisfies every member of Γ with s, $\mathfrak{A}$ also satisfies ϕ with s.
Further, we define ϕ and ψ to be *logically equivalent* iff $\phi \models \psi$ and $\psi \models \phi$.

The first-order analogue of the concept of a tautology is the concept of a valid formula: A wff ϕ is *valid* iff $\emptyset \models \phi$ (written $\models \phi$). Thus ϕ is valid iff for every $\mathfrak{A}$ and every $s : V \to |\mathfrak{A}|$, $\mathfrak{A}$ satisfies ϕ with s.

Corollary. For a set Σ and τ of sentences, $\Sigma \models \tau$ iff every model of Σ is also a model of τ. A sentence τ is valid iff it is true in every structure.

Next, Enderton devlops the notion of *definability* in a structure, that is, what is a given structure capable of defining? A good example is the case of the real field, given by $\mathfrak{R} = (\mathbb{R};\ 0,\ 1,\ +,\ \times)$. In $\mathfrak{R}$, what numbers could we define? What set of numbers can be defined in $|\mathfrak{R}|$ using only first-order logic and the other symbols provided?
Consider the sentence σ, where $\sigma = \exists v_2(x = v_2 \times v_2)$. σ then defines the set of all positive numbers, but leaves out all negative numbers. Thus, we can say that σ defines the interval $[0, \infty)$ in $\mathfrak{R}$.

Now, we want to be able to determine what kind of models (what *class* of models) will satisfy a given set of sentences. Consider a given set Σ of sentences. $Mod\ \Sigma$ is then defined as the class of all structures where each structure satisfies each member of Σ. We say that $Mod\ \Sigma$ defines a *class* of structures, as opposed to a *set*, because the number of structures is too large to be a set and is a proper class (as long as $Mod\ \Sigma$ is nonempty, of course).
We say that a class κ of structures is an *elementary class (EC)* iff $\kappa = Mod\ \tau$ for some sentence τ. Also, κ is an *elementary class in the wider sense (EC_δ)* iff $\kappa = Mod\ \Sigma$ for some set Σ of sentences.

Dealing with so many structures, is it possible for any two distinct structures to be "equivalent"? And if so, how? Here is where Enderton develops a notion equivalence among structures.

Definition (Isomorphic Structures). Two structure, $\mathfrak{A}$ and $\mathfrak{B}$, are said to be *isomorphic* iff there is a one-to-one correspondence betwen their universes, $|\mathfrak{A}|$ and $|\mathfrak{B}|$, that preserves the operations and relations.

To formally define this notion and to show that two isomorphic structures must satisfy exactly the same sentences, we introduce the notion of a *homomorphism*.

Definition (Homomorphism & Isomorphism). Let $\mathfrak{A}$ and $\mathfrak{B}$ be two distinct structures. A *homomorphism* h of $\mathfrak{A}$ into $\mathfrak{B}$ is a function $h : |\mathfrak{A}| \to |\mathfrak{B}|$ with the properties:

1. For each n-place predicate parameter P and each n-tuple $< a_1, \ldots, a_n >$ of elements of $|\mathfrak{A}|$,

$$< a_1, \ldots, a_n > \in P^{\mathfrak{A}} \quad \text{iff} \quad < h(a_1), \ldots, h(a_n) > \in P^{\mathfrak{B}} \tag{5}$$

2. For each n-place function symbol f and each n-tuple,

$$h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n)) \tag{6}$$

For any constant symbols, we simply have

$$h(c^{\mathfrak{A}} = c^{\mathfrak{B}} \tag{7}$$

If h is one-to-one, then it is called an *isomorphism.*

Theorem (Homomorphism Theorem). *Let h be a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, and let s map the set of variables into $|\mathfrak{A}|$. Then,*

1. *For any term t, we have $h(\overline{s}(t)) = \overline{h \circ s}(t)$, where $\overline{s}(t)$ is computed in $\mathfrak{A}$ and $\overline{h \circ s}(t)$ is computed in $\mathfrak{B}$.*
2. *For any quantifier-free formula α not containing the equality symbol,*

$$\models_{\mathfrak{A}} \alpha[s] \quad \textit{iff} \models_{\mathfrak{B}} \alpha[h \circ s] \tag{8}$$

3. *If h is one-to-one then in part (2), we may delete the restriction that α not contain the equality symbol.*
4. *If h is a homomorphism of* $\mathfrak{A}$onto $\mathfrak{B}$*, then in part (2) we may delete the restriction that α be quantifier-free.*

Definition (Elementarily Equivalent Structures). To structures $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *elementarily equivalent* (written $\mathfrak{A} \equiv \mathfrak{B}$) iff for any sentence σ,

$$\models_{\mathfrak{A}} \sigma \quad \leftrightarrow \quad \models_{\mathfrak{B}} \sigma \tag{9}$$

Corollary. Isomorphic structures are elementarily equivalent. That is,

$$\mathfrak{A} \cong \mathfrak{B} \quad \Rightarrow \quad \mathfrak{A} \equiv \mathfrak{B} \tag{10}$$

Corollary. Let h be an automorphism of the structure $\mathfrak{A}$, and let R be an n-ary relation on $|\mathfrak{A}|$ definable in $\mathfrak{A}$. Then for any $a_1, \ldots, a_n$ in $|\mathfrak{A}|$,

$$< a_1, \ldots, a_n > \in R \quad \leftrightarrow \quad < h(a_1), \ldots, h(a_n) > \in R \tag{11}$$

2.6 Models of Theories

In a general sense, there are two types of models : finite and infinite. Some sentences may have only finite models while others have only infinite. Some may have both. An example is the sentence stating that $<$ is an ordering relation with no largest element. Clearly, this statement is only true in infinite models. Its negation, however, is true in any finite model (and is thus called *finitely valid*). For another good example, consider the sentence $\forall x \forall y(x = y)$. This sentence is true in only finite models with one element!

Theorem. *If a set Σ of sentences has arbitrarily large finite models, then it has an infinite model.*

The above theorem is important. It states that it is impossible for any set of sentences to be true in every finite model and false in every infinite one.

Corollary. The class of all finite structures is not EC_δ (for any set of sentences). The class of all infinite structures is not EC (for any single sentence).

Previously, we defined $Mod\ \Sigma$ to be the class of all models that satisfied a set of sentences. Now, we want to do a similar opertion only on models. Let *the theory of* $\mathfrak{A}$, written $Th\ \mathfrak{A}$, be defined as the set of all sentences true in $\mathfrak{A}$.

Theorem. *For a finite structure $\mathfrak{A}$, $Th\ \mathfrak{A}$ is decidable.*

Theorem. *For a finite language, $\sigma | \sigma sigma$ has a finite model is effectively enumerable.*

Corollary. For a finite language, let Φ be the set of sentences true in every finite structure. Then its complement, $\overline{\Phi}$, is effectively enumerable.

Theorem (Trakhtenbrot's Theorem). *The set of sentences*

$$\Phi = \{\sigma | \sigma \textit{is true in every finite structure}\} \tag{12}$$

is not in general decidable or effectively enumerable.

Next, Enderton discusses the size of models. It can be shown that a consistent set of sentences in a countable language has a countable model. This leads us to the Löwenheim-Skolem theorem:

Theorem (Löwenheim-Skolem Theorem - Countable Models Only). *1. Let Γ be a satisfiable set of formulas in a countable language. Then Γ is satisfiable in some countable structure.*

2. Let Σ be a set of sentences in a countable language. If Σ has any model, then it has a countable model.

This theorem can be applied to prove the following. Choose any consistent set of axioms for set theory. By this theorem, the axioms have some countable model, $\mathfrak{G}$. We know from previous theorems that since $\mathfrak{G}$ is a model of the chosen axioms, it must also be a model of all sentences logically implied by those axioms. Here, an interesting paradox (known as Skolem's paradox) arises. Of all the sentences logically implied by these axioms, one states that there are uncountably many sets. Enderton assures us that there is no contradiction here, since it is true that in $\mathfrak{G}$ there can be a point (set) that cannot be put in a one-to-one mapping with the natural numbers, as long as the *number* of points (sets) is countable.

The Löwenheim-Skolem theorem can also be applied to show that for any structure $\mathfrak{A}$ for a countable language, there is a countable structure $\mathfrak{B}$ that is elementarily equivalent to $\mathfrak{A}$ ($\mathfrak{A} \equiv \mathfrak{B}$).

Concerning uncountable languages, the Löwenheim-Skolem theorem can be stated as follows:

Theorem (Löwenheim-Skolem Theorem). *1. Let Γ be a satisfiable set of formulas in a language of cardinality λ. Then Γ is satisfiable in some structure of size $\leq \lambda$.*

2. Let Σ be a set of sentences in a language of cardinality λ. If Σ has any model, then it has a model of cardinality $\leq \lambda$.

The earlier version of this theorem can be stated as a special case, where $\lambda = \aleph_0$.

Theorem (LST Theorem). *Let Γ be a set of formulas in a language of cardinality λ, and assume that Γ is satisfiable in some infinite structure. Then for every cardinal $\kappa \leq \lambda$, there is a structure of cardinality κ in which Γ is satisfiable.*

Corollary. 1. Let Σ be a set of sentences in a countable language. If Σ has some infinite model, then Σ has models of every infinite cardinality.

2. Let $\mathfrak{A}$ be an infinite structure for a countable language. Then for any infinite cardinal λ, there is a structure $\mathfrak{B}$ of cardinality λ such that $\mathfrak{B} \equiv \mathfrak{A}$.

The above corollary can be used to show the following. Call a set Σ of sentences *categorical* iff any two models of Σ are isomorphic. The above corollary implies that if Σ has any infinite models, then Σ is not categorical. For example, there is no set of sentences whose models are exactly the structures isomorphic to $(\mathbb{N};\ 0,\ S,\ +,\ \cdot)$. Thus, first-order languages are limited in their expressive powers.

Next, Enderton provides a new definition for a *theory*. Where we previously defined a theory to be the set of all sentences implied by a smaller set of sentences (thus, it was defined in relation to a set of sentences), we now define a theory to be a set of sentences closed under logical implication. (It should be easy to see the similarity between the two definitions). With this new definition, T is a theory iff T is a set of sentences such that for any sentence σ of the language,

$$T \vDash \sigma \rightarrow \sigma \in T \tag{13}$$

Definition (Axiomatizable). A theory T is *axiomatizable* iff there is a decidable set of Σ of sentences such that $T = Cn\Sigma$.

Definition (Finitely Axiomatizable). A theory T is *finitely axiomatizable* iff $T = Cn\Sigma$ for some finite set Σ of sentences.

Theorem. *If $Cn\Sigma$ is finitely axiomatizable, then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $Cn\Sigma_0 = Cn\Sigma$.*

Corollary. 1. An axiomatizable theory (in a reasonable language) is effectively enumerable.

2. A complete axiomatizable theory (in a reasonable language) is decidable.

We say that a theory T is $\aleph_0$-categorical iff all the infinite countable models of T are isomorphic. More generally, for a cardinal κ, say that T is κ-categorical iff all models of T having cardinality κ are isomorphic.

Theorem (Loś-Vaught Test). *Let T be a theory in a countable language. Assume that T has no finite models.*

1. *If T is $\aleph_0$-categorical, then T is complete.*
2. *If T is κ-categorical for some infinite cardinal κ, then T s complete.*

Theorem. *1. The theory of algebraically closed fields of characteristic 0 is complete.*

2. The theory of the complex field $\mathfrak{C} = (\mathbb{C};\ 0,\ 1,\ +,\ \cdot)$ is decidable.

Theorem. *Any countable model of δ is isomorphic to $(\mathbb{Q}, <_Q)$.*

Definition (Prenex Normal Form). A formula is said to be in *prenex normal form* iff for some $n \geq 0$

$$Q_1 x_1 \dots Q_n x_n \alpha \tag{14}$$

where Q_i is $\forall$ or $\exists$ and α is quantifier-free.

Theorem (Prenex Normal Form Theorem). *For any formula, we can find a logically equivalent formula in prenex normal form.*

4 Second-Order Logic

4.1 Second-Order Languages

coming soon!

4.2 Skolem Functions

coming soon!

4.3 Many-Sorted Logic

coming soon!

4.4 General Structures

coming soon!