The Independence of the General Continuum Hypothesis

Ryan Flannery

1 Introduction

In this paper, the independence of the generalized continuum hypothesis of the Zermelo-Fraenkel set-theory axioms will be discussed in a manner that individuals with only a mild background in set theory can follow. This is not, however, an exhaustive trace through the details of the proof. Rather, most of details will be omitted.

NOTE: Currently, this paper only covers Gödel's half of the proof, stating that ZF set theory cannot disprove GCH. The other half of the proof by Cohen, that ZF cannot prove GCH, will (hopefully) be filled in later (I make no promises as to when).

This paper is divided into four sections. The first section (this section) briefly reviews topics that will be required for the rest of the paper. The second section covers the proof of Gödel's theorem that the generalized continuum hypothesis cannot be *disproved* in Zermelo-Fraenkel set theory. The third section simply states the proof of Cohen's theorem that the generalized continuum hypothesis cannot be *proved* in Zermelo-Fraenkel set theory. The final section uses these two theorems to conclude that the generalized continuum hypothesis is *undecidable* in Zermelo-Fraenkel set theory.

1.1 The Zermelo-Fraenkel Axioms

The following set of axioms were constructed by by Ernst Zermelo, Adolf Fraenkel, and Thoralf Skolem for the purpose of axiomatizing set theory, and are considered the standard axioms for set theory. The axiom of choice is omitted for the purpose of this paper.

The language of Zermelo-Fraenkel set theory is very close to that of first-order logic. In fact, it contains all of first-order logic with the addition of a new relation symbol '$\in$', which is interpreted in the following manner:

$$x \in y \text{ means that } x \text{ is a member of the set } y$$

Axiom of Extensionality. Two sets are equal if and only if they have identical elements. In the language of set theory, two sets x and y are equal if and only if:

$$\forall z(z \in x \leftrightarrow z \in y)$$

Null-Set Axiom. There exists a set with no members (a.k.a. the "empty set", and denoted by $\emptyset$). In the language of set theory, the $\emptyset$ can be expressed as the set y in the sentence,

$$\forall x(\neg(x \in y))$$

Axiom of Infinity. There exists a set x such that $\emptyset \in x$ and such whenever $a \in x$, $\{a\} \in x$ also.[1] In the language of set theory,

$$\exists x(\emptyset \in x \wedge \forall a(a \in x \leftrightarrow \{a\} \in x))$$

Power Set Axiom. If x is a set, then there is a set consisting of all and only the subsets of x. In the language of set theory, the power set of x can be expressed as the set y in the sentence

$$\forall x \exists y \forall z(z \in y \leftrightarrow \forall a(a \in z \rightarrow a \in x))$$

Axiom of Union. If x is a set then there is a set whose members are precisely the members of the members of x (often denoted by $\bigcup x$). In the language of set theory,

$$\forall x \exists y \forall z(z \in x \leftrightarrow \forall a(a \in z \leftrightarrow a \in y))$$

Axiom of Replacement. Let $\phi(x, y)$ be any formula in the language of set theory such that for every set α there is a unique set β satisfying $\phi(\alpha, \beta)$. Let x be any set. Then there exists a set y consisting of just those b such that $\phi(a, b)$ for some $a \in x$. Notice that since there is no way of working with whole formulas in the language of set theory, this is really an axiom schema given a set of formula's for ϕ.

Axiom of Subset Selection. (a.k.a. the Axiom of Separation) Given any set x and a proposition $\phi(y)$, there exists a subset of x containing only the elements where $\phi(y)$ holds. As in the Axiom of Replacement, this requires an axiom schema to express in the language of set theory.

Axiom of Foundation. (Sometimes called the Axiom of Regularity.) For every nonempty set x, there is a set $a \in x$ such that $a \cap x = \emptyset$. Notice, this is equivalent to stating that $\in$ is a well-founded relation. In the language of set theory,

$$\forall x(x \neq \emptyset \rightarrow \exists a(a \in x \wedge a \cap x = \emptyset))$$

[1] Note that this axiom guarantees the existence of an infinite set. Since $\emptyset \in x$, so too does $\{\emptyset\} \in x$, and $\{\{\emptyset\}\} \in x$, and so on...

Collectively, these set of axioms of the collections of theorems they prove are called ZF set theory, and this abbreviation will be used for the remainder of this paper.

Within these axioms it is possible to define the universe of sets, usually denoted V. The null-set axiom asserts that the empty set ($\emptyset$) exists, and the power set axiom can be used to build successive levels in this hierarchy. For example,

Level		**Set**	
V_0	$=$	$\emptyset$	By the null-set axiom.
V_1	$= P(V_0) =$	$\{\emptyset\}$	By the power set axiom.
V_2	$= P(V_1) =$	$\{\emptyset, \{\emptyset\}\}$	By the power set axiom.
V_3	$= P(V_2) =$	$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$	By the power set axiom.
$\vdots$		$\vdots$	

where $P(x)$ denotes the power set of a set x.

This isn't quite the entire universe of sets, however. The axiom of infinity asserts there exists at least one infinite set, and although the V hierarchy defined above grows exponentially, the axiom of infinity is needed to assert the existence of infinite sets. For an infinite set of cardinality α (α is a limit ordinal), V_α is defined as the union of all lesser levels V_i of the hierarchy. This union exists by the axiom of union, and

$$V_\alpha = \bigcup_{i<\alpha} V_i$$

The universe V is then defined as the union of all levels V_i of this hierarchy, and this universe constitutes the domain which $\forall$ and $\exists$ quantify over in the language of ZF set theory.

NOTE: The above definition of V_α, if taken literally, is an infinitely long formula. *Such a formula is not allowed in set theory!* The set V_α is actually defined using the axiom of replacement. The "$\bigcup$" notation is used because it conveys the meaning more clearly than the actual formula.

1.2 The Axiom of Constructibility

The following axiom that is discussed is not a standard part of ZF set theory. However, it plays a critical role in Gödel's proof of the consistency of GCH with ZF.

In the ZF axioms listed in Section 1.1, it may be noticed that nowhere was a set formally defined. As such, the notion of subset and power set are quite vague. Indeed, the version of power set adopted in Section 1.1 is often referred to as the *unrestricted power set* (the meaning of this name will become clear in a moment). The idea behind the axiom of constructibility is to remove this vagueness, and formalize the notion of a set, a subset, and the power set of a set.

First, let a set be any *describable* collection, where "describable" is meant as "definable in the language of set theory" (which is first-order logic along with the '$\in$' relation). Next, replace the previous definition of the power set operation with a "describable power set" operation (or a *restricted power set* operation). With this new power set operation, the power set of a set x is the set of all describable subsets of x. Finally, we build up the universe of sets just as was before only using this new notion of power set. The resulting universe is called the *constructible hierarchy*, and usually denoted by L.

Level		**Set**	
L_0	$=$	$\emptyset$	By the null-set axiom.
L_1	$= P(L_0) =$	$\{\emptyset\}$	By the new power set axiom.
L_2	$= P(L_1) =$	$\{\emptyset, \{\emptyset\}\}$	By the new power set axiom.
L_3	$= P(L_2) =$	$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$	By the new power set axiom.
$\vdots$		$\vdots$	

And for a limit ordinal α,

$$L_\alpha = \bigcup_{i<\alpha} L_i$$

This may appear to be identical to the V hierarchy, but the describable power set operation does in fact produce a different hierarchy. Notice, for a given level L_α, $a \in L_{\alpha+1}$ if and only if there is some formula $\phi(x)$ in the language of set theory with one free variable, x, such that the elements of a satisfy $\phi(x)$[2]. For infinite sets, this is where the L hierarchy separates and the V hierarchy separate. Namely, L becomes a sub-hierarchy of V, since for each level of the hierarchy L_α is contained in V_α.

When adopting the axiom of constructibility, the universe of sets is considered to be precisely L, and replaces V. Thus, the usual notation for ZF set theory with the axiom of constructibility is $ZF + (V = L)$, and this notation is adopted for the remainder of this paper.

One important theorem in ZF $+(V = L)$ is that L can be well-ordered, even if the axiom of choice is not adopted. This fact will also play a role in Gödel's proof of the relative consistency of GCH in ZF set theory.

1.3 The Continuum Hypothesis

The continuum hypothesis is a statement first formulated by George Cantor regarding the sizes of infinite sets. Cantor had shown that the set of natural numbers $\mathbb{N}$ was smaller than the set of real number $\mathbb{R}$. Using the notion of cardinality (degrees of infinity) introduced by Cantor, the cardinality of $\mathbb{N}$ is less than the cardinality of $\mathbb{R}$. Cantor's original statement of the continuum hypothesis is the following:

[2]Here, the formula ϕ may contain quantifiers. If it does, keep in mind that they do not quantify over all of L since that has not been constructed yet. Rather, they quantify over the previous levels of the hierarchy.

There is no set whose size is strictly between that of $\mathbb{N}$ *and that of* $\mathbb{R}$.[3]

In modern notation, the cardinality of $\mathbb{N}$ is $\aleph_0$ and the cardinality of $\mathbb{R}$ is $\aleph_1$. Using this notation, the continuum hypothesis is:

$$\neg\exists a(\aleph_0 < |a| < \aleph_1)$$

where $|a|$ denotes the cardinality of the set a. An equivalent statement is that

$$2^{\aleph_0} = \aleph_1.$$

The generalized continuum hypothesis (abbreviated from here on as GCH) is, as it sounds, a generalization of the continuum hypothesis. It states that for any infinite set of cardinality α, the power set of this set has cardinality $\alpha + 1$, and there are no sets with cardinality greater than α but less than $\alpha + 1$. Formally,

$$\neg\exists a(\aleph_\kappa < |a| < \aleph_{\kappa+1})$$

or,

$$2^{\aleph_\kappa} = \aleph_{\kappa+1}$$

for κ a cardinal.

There is, however, other notation used for stating CH and GCH. For an infinite set s with cardinality α, let s^* denote a set with cardinality $\alpha + 1$ (the next cardinality). Using this notation, the generalized continuum hypothesis simply states that $P(s) = s*$, where s is an infinite set. The continuum hypothesis is simply a special case where s has cardinality $\aleph_0$, and s^* has cardinality $\aleph_1$.

Cantor was able to prove that the for any infinite set s, $P(s)$ has cardinality of *at least* s^*. This follows since Cantor showed that $P(s)$ has greater cardinality than s, and s^* is the least cardinality greater than s. What Gödel shows, is that for a set s, $P(s)$ has cardinality of *at most* s^*.

1.4 Undecidability

As a result of Gödel's first incompleteness theorem, any formal system capable of representing the natural numbers is *incomplete*, meaning there are statements in that system that cannot be proved true or false. Such statements are considered *undecidable* since their truth or falsity cannot be ascertained. Formally, the notation $\Sigma \vdash \phi$ is used to express that a sentence ϕ is provable from a set of sentences Σ (usually a set of axioms). In this notation, a statement ϕ is undecidable from a set of sentences Σ if and only if

$$\Sigma \nvdash \phi$$

and

$$\Sigma \nvdash \neg\phi.$$

Note that $\Sigma \nvdash \neg\phi$ is also required for undecidability. If this were not the case, and $\Sigma \vdash \neg\phi$, then it is simply the case that ϕ is false, not undecidable.

For the purpose of this paper, the set of sentences Σ will always be either the ZF axioms or ZF $+(V = L)$.

[3] The set of real numbers $\mathbb{R}$ is also called the *continuum*. Hence the name of this hypothesis.

1.5 Consistency of Sentences

This section covers a theorem in mathematical logic that allows Gödel to prove the relative consistency of GCH. Formally stated, the theorem states the following:

> Let Σ be a consistent set of sentences. If Σ plus some additional sentence σ is also consistent, and if $\Sigma + \sigma \vdash \phi$, then it is impossible for $\Sigma \vdash \neg\phi$. More succinctly, *if Σ and $\Sigma + \sigma$ are consistent, then*
>
> $$\Sigma + \sigma \vdash \phi \rightarrow \Sigma \nvdash \neg\phi.$$

Given a Σ, σ, and ϕ as described above, knowing that $\Sigma + \sigma \vdash \phi$ only allows $\Sigma \nvdash \neg\phi$ to be inferred. It does *not* allow $\Sigma \vdash \phi$ to be inferred.

By showing that $\Sigma \nvdash \neg\phi$, one can "safely adopt" ϕ without introducing any contradictions since it is impossible for Σ to prove $\neg\phi$. In such a case, it said that ϕ is consistent with Σ.

2 Gödel's Proof

In 1940, Kurt Gödel proved that GCH cannot be disproved from the ZF axioms, even if the axiom of choice is included. Specifically, Gödel's theorem states the following:

If ZF is consistent then it remains consistent if GCH is added.

Which is equivalent to both of the following statements:

- If ZF is consistent then GCH is not disprovable.
- If ZF is consistent then $\neg GCH$ is not provable.

This is proved in the following manner. We first consider the non-standard axiom of constructibility ($(V = L)$) and add it to our list of axioms for set theory (thus, we are working with ZF $+(V = L)$). First, Gödel shows that if ZF is consistent then ZF $+(V = L)$ is also consistent. Next, Gödel proves GCH from ZF $+(V = L)$. He then uses the result from mathematical logic discussed in Section 1.5 to conclude that $ZF \nvdash \neg GCH$.

The detailed proof below is broken into two parts. First, the proof that if ZF is consistent then ZF $+(V = L)$ is consistent is discussed. Then, the proof that ZF $+(V = L)$ proves GCH is discussed.

2.1 The Relative Consistency of ZF $+(V = L)$

As was repeatedly stated above, a proof of GCH from ZF $+(V = L)$ is useful only if ZF $+(V = L)$ is consistent. If it were not the case that ZF $+(V = L)$ were consistent, then anything would be provable (via a contradiction), and the system wouldn't be sound. So, it needs to be shown that ZF $+(V = L)$ is consistent.

However, as a result of Gödel's second incompleteness theorem and since ZF is capable of defining the concept of natural numbers, if ZF is consistent then it is *impossible* to prove so within ZF itself. As such, proving the consistency of ZF $+(V = L)$ (within ZF $+(V = L)$) is also impossible (indeed, proving the consistency of $ZF + Anything$ is impossible, unless it is inconsistent).

Instead, the *relative consistency* of ZF $+(V = L)$ is proved. That is, the consistency of ZF is simply accepted, and it is then shown that ZF $+(V = L)$ *must* also be consistent.

$$ZF \text{ is Consistent} \rightarrow ZF + (V = L) \text{ is Consistent}$$

Initially, one might think that adding an additional axiom to ZF may make the system more susceptible to being inconsistent, since $(V = L)$ may contradict the other axioms. Gödel shows, however, that if ZF is consistent it remains so when $(V = L)$ is accepted. He does so in the following manner.

Consider any arbitrary model $\mathfrak{A}$ of ZF. Since we accept that ZF is consistent, such a model must exist. In Section 1.1 it was shown that by using the ZF axioms, one could construct the V hierarchy. Now, in Section 1.2 a more strict version of the power set operation was defined, and the L (constructible) hierarchy was constructed. It was briefly mentioned that the for corresponding levels of the L and V hierarchy's, L is always a sub-hierarchy of V since it is *restricted* in the way it grows using the power set operation. As such, $\mathfrak{A}$ *must contain a submodel that is a model of ZF +(*$V = L$*) !* Thus, any model of ZF also constitutes a model of ZF $+(V = L)$. That is, for all models $\mathfrak{A}$,

$$\models_{\mathfrak{A}} ZF \rightarrow \models_{\mathfrak{A}} ZF + (V = L)$$

where $\models_{\mathfrak{A}} \Sigma$ is read as "$\mathfrak{A}$ is a model of Σ".

As such, in any interpretation where ZF is consistent, ZF $+(V = L)$ is automatically consistent, and it is said that *ZF +(*$V = L$*) is consistent relative to ZF.*

2.2 $ZF + (V = L) \vdash GCH$

Perhaps the reason by $ZF \not\vdash GCH$ is because in ZF set theory, the notion of set was never formally defined. Also, the power set operation was *unrestricted.* That is, $V_{\alpha+1}$ is defined as $P(V_\alpha)$, which is to say that $V_{\alpha+1}$ is the set of all subsets of V_α. Without a precise definition of a set, there is no precise definition of a subset, and therefore no precise definition for the power set of a set. How can one prove any statement regarding the relative sizes of a set and its power set (which is exactly what GCH states) without such precision?

As discussed in Section 1.2, by adopting the axiom of constructibility, an entirely different hierarchy is constructed, called the *constructible hierarchy* (or L) in which sets have a precise definition. This definition also gives a precise definition of a *subset* and the *power set* of a set.

Let S be any set. In this case, let S be infinite. Then there exists a functions ϕ with one free variable in normal form[4], such that $\phi(x)$ defines a set S' containing all subsets of S. Roughly, $\phi(x)$ then looks something like the following:

$$\forall x_1, \ldots, x_a \exists y_1, \ldots, y_b \forall z_1, \ldots, z_c \exists u_1, \ldots, u_d \ldots$$
$$L(\ldots x \ldots x_1 \ldots x_a \ldots y_1 \ldots y_b \ldots z_1 \ldots z_c \ldots u_1 \ldots u_d)^5$$

Let f be a function in S. That is, f has n variables $(x_1, \ldots, x_n)$ and for any elements $s_1, \ldots, s_n$ of S, $f(s_1, \ldots, s_n)$ defines an element of S.

Next, *Skolemize* ϕ, by replacing all universally quantified variables with Skolem constants (for this example, the name of the constants are left as the name of the variables) and all existentially quantified variables with Skolem functions $f_1, \ldots, f_m$, where each f_i is a function in S.

$$L(\ldots x \ldots x_1 \ldots x_a \ldots f_1(x_1, \ldots, x_a) \ldots f_b(x_1, \ldots, x_a) \ldots z_1 \ldots z_c \ldots$$
$$f_k(x_1, \ldots, x_a, z_1, \ldots z_c) \ldots f_{k+d}(x_1, \ldots, x_a, z_1, \ldots z_c) \ldots$$

Using this approach, it is possible to construct functions defining each level of the constructible hierarchy. By definition of the L hierarchy, if two ordinals α and β are such that $\alpha < \beta$, then it's easily seen that $L_\alpha \in L_\beta$ and $L_\alpha \subset L_\beta$. Further, it can be proven that for any sets x and y, if $x \in y$ then the size (or cardinality in the infinite case) of x is less than the size (cardinality) of y.

This, in the constructible hierarchy, proves for any constructible subset s of a set L_α (α an infinite ordinal), s *must* have a cardinality less than α. An equivalent (and more relevant) statement of this is that for a set S all of whose subsets have cardinality less than α, the cardinality of S must be less than $\alpha+1$. Now, consider any infinite cardinal α, and its associated level of the L hierarchy L_α. This means that every (constructible) subset of L_α is an element of L_{α^*}, and that the cardinality of L_{α^*} must be less than the cardinality of $L_{\alpha^{**}}$. That is, the cardinality of $P(S)$ is at most the cardinality of S^*.

As discussed in Section 1.3, Cantor proved

$$|P(x)| \geq |x^*|$$

and Gödel just showed that

$$|P(x)| \leq |x^*|.$$

Thus, it must be the case that

$$|P(x)| = |x^*|$$

which is the statement of the generalized continuum hypothesis.

[4]By *normal form*, it is meant that ϕ contains variables $(x_1, x_2, \ldots)$, constants $(c_1, c_2, \ldots)$, the '$\in$' relation symbol, $\neg$ and $\vee$, and quantifiers ($\forall$ and $\exists$) which quantify all variables over S.

[5]The ordering of variables in L is arbitrary. They may be arranged in whatever order is needed.

2.3 Gödel's Conclusion

All of the pieces are now present to prove that GCH is consistent with ZF. By accepting that ZF is consistent, it was shown that ZF $+(V = L)$ is also consistent. Then, it was shown that $ZF + (V = L) \vdash GCH$.

1	ZF is Consistent.	Premise
2	$ZF + (V = L)$ is Consistent.	by Section 2.1
3	$ZF + (V = L) \vdash GCH$	by Section 2.2
4	$ZF \nvdash \neg GCH$	by Section 1.5

Thus, as long as ZF is consistent, $ZF \nvdash \neg GCH$, or

GCH is relatively consistent with ZF set theory.

3 Cohen's Proof

In 1963, Paul Cohen proved that GCH cannot be proven from the ZF axioms. Although Cohen's proof is not discussed in this paper, it is the second piece of the proof required for proving the independence of GCH from the ZF axioms.

Formally, Cohen proves the following:

$$ZF \nvdash GCH$$

4 Conclusion & Remarks

By Gödel's proof,

$$ZF \nvdash \neg GCH,$$

and by Cohen's proof,

$$ZF \nvdash GCH.$$

In other words, within ZF set theory it is impossible to prove either GCH or $\neg GCH$. Thus, GCH is considered *undecidable*, since it's truth or falsity cannot be ascertained.

With this result, one may wonder why GCH isn't simply adopted as an additional axiom for set theory. There are two problems with this.

1. *GCH isn't intuitive.* Although for finite sets it is easy to see that for a set x, the $P(x)$ contains precisely $2^{|x|}$ elements, it isn't clear for case that x is infinite. As such, GCH has no place as an axiom.

2. *Adopting GCH still leaves some problems undecidable.* For example, Whitehead's problem with abelian groups and Souslin's problem of Dedekind complete sets with no end points still remain undecidable (neither of these problems are discussed in detail here). By adopting GCH as an axiom, very little is gained.

Ideally, one would want to find some other axiom to adopt which is intuitive and, like the axiom of constructibility, proves GCH. Although the axiom of constructibility proves GCH, it (like GCH) isn't intuitive.

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