Mathematical Logic III
The Joy of Sets
§2.1 - §2.4
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2 The Zermelo-Fraenkel Axioms

In chapter 2, Devlin introduces a formal language for set theory as well as a set of axioms that will provide a framework for set theory in general. The axioms, known as the Zermelo-Fraenkel axioms, are all intuitive in nature and are arguably self-evident 'truths' about sets. Since set theory provides a sort of 'foundation' for mathematics as a whole, it's certainly desirable to have something so powerful rest on solid ground.

The language introduced is restrictive in that it only allows the construction of mathematically oriented collections, but general enough to allow the construction of *any* mathematically oriented collection.

2.1 The Language of Set Theory

The language we use for set theory is precise in both its notation and grammar, making it a formal language. We abbreviate this language with the name LAST (the LAnguage of Set Theory.)

When referring to a specific set in LAST we shall always use the name w. When referring to more than one specific set, we shall always use the name w with integer subscripts (starting with 0) for our names (i.e. $w_0, w_1, w_2, \ldots$)

Similarly, when referring to arbitrary sets or variables (remember: *everything is a set!*) we shall always use the name v with integer subscripts (starting with 0) for our names (i.e. $v_0, v_1, v_2, \ldots$)

As for symbols, we inherit all of the standard logical symbols from first order logic along with the membership symbol, $\in$. Thus, a complete lexicon for LAST is:

Symbol	Meaning	Symbol	Meaning
$w_0, w_1, \ldots$	Named Sets	$v_0, v_1, \ldots$	Variable Sets
$\in$	Membership	$=$	Equality
$\wedge$	Conjunction	$\vee$	Disjunction
$\neg$	Negation	(,)	Parentheses (for grouping)
$\forall$	Universal Quantifier	$\exists$	Existential Quantifier

As for the syntax of LAST, the following and *only the following* represent valid formulas (be they clauses, phrases, or sentences):

1. Any expression of the following form is a valid formula

$$(v_n = v_m) \quad (v_n = w_m) \quad (w_m = v_n) \quad (w_n = w_m)$$
$$(v_n \in v_m) \quad (v_n \in w_m) \quad (w_m \in v_n) \quad (w_n \in w_m)$$

2. If ϕ and ψ are valid formulas, so too are

$$(\phi \ \wedge \ \psi) \quad (\phi \ \vee \ \psi)$$

3. If ϕ is a valid formula, so too is

$$(\neg\phi)$$

4. If ϕ is a valid formula, so too are

$$(\forall v_n \phi) \quad (\exists v_n \phi)$$

The above for methods are *the only* valid methods we allow for constructing formulas in LAST.

Remember, any formula with no free variables is a *sentence*, and sentences can be evaluated to true or false. Since formulas contain one or more free variables, which are *arbitrary* sets, formulas can *not* be assigned a definite truth value.

When dealing with formulas that have free variables, we often use the abbreviation $\phi(v_0, \ldots, v_n)$ to indicate that ϕ is a formula that has free variables in the list $(v_0, \ldots, v_n)$. With this, we can describe *collections* of sets in LAST. For example, suppose we have a formula $\phi(v_n)$ in LAST and we know what all of the specified sets in the formula (those with names like w_n) refer to. Then, for any set x, we can determine if $\phi(x)$ is true or false, and we can define a *collection* as all of the sets that satisfy $\phi(v_n)$.

For our own ease, we also allow the following abbreviations in LAST:

Abbreviation	Meaning
$(\phi \rightarrow \psi)$	$((\neg\phi) \vee \psi)$
$(\phi \leftrightarrow \psi)$	$((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$
$x \subseteq y$	$(\forall v_n((v_n \in x) \rightarrow (v_n \in y)))$
$x = \bigcup y$	$(\forall v_n((v_n \in x) \leftrightarrow \exists v_m((v_n \in v_m) \wedge (v_m \in y))))$ *where* $n \neq m$ *and* $v_n, v_m \neq x, y$
$x = \{y\}$	$(\forall v_n((v_n \in x) \leftrightarrow (v_n = y)))$ *where* $v_n \neq x, y$
$x = \{y, z\}$	$(\forall v_n((v_n \in x) \leftrightarrow ((v_n = y) \vee (v_n = z))))$ *where* $v_n \neq x, y, z$
$x = (y, z)$	$(\forall v_n((v_n \in x) \leftrightarrow ((v_n = \{y\}) \vee (v_n = \{y, z\}))))$ *where* $v_n \neq x, y, z$
$x = y \cup z$	$x = \bigcup\{y, z\}$

With these abbreviations, using LAST to express the concepts of set theory will be a bit less cumbersome. However it is important to note that any concept expressed using these abbreviations could just as easily be expressed without the abbreviations by simply substituting the abbreviation with its meaning.

Exercise 2.1.1. Define the following.

(i) x is an ordered pair

Answer. As given in the text...

$$\exists v_n(\exists v_m(x = (v_n, v_m)))$$

(ii) x is a function

Answer. From page 13 of the text, a function on a set is a relation (a set of ordered pairs) such that for every a in the domain, there is exactly one b in the range such that the ordered pair (a, b) is in the relation.

$$\forall a \exists b((a, b) \in x \rightarrow \forall c((a, c) \in x \leftrightarrow c = b))$$

(iii) $x = y \times z$

Answer. From page 10 of the text, the *cartesian product* of x and y is defined as the set of all ordered pairs (a, b) such that $a \in x$ and $b \in y$.

$$\forall a \forall b((a, b) \in x \leftrightarrow (a \in y \wedge b \in z))$$

(iv) x is an n-ary function from y to z

Answer. From page 13 of the text, an n-ary function x on a set y is defined as an $(n + 1)$-ary relation on y such that for every a in the domain of the relation there is exactly one b in the range such that (a, b) is in the relation.

$$\forall_{a \in y_1 \times \ldots \times y_n} \exists_{b \in y}((a, b) \in x \rightarrow \forall_{c \in y}((a, c) \in x \leftrightarrow c = b))$$

(v) x is a poset

Answer. $x = (y, z)$ is a poset iff z is a binary relation on y that is reflexive, antisymmetric, and transitive.

$$\begin{aligned}\exists y \exists z((y, z) = x \wedge \quad & ((z = y \times y) \\ & \wedge (\forall_{a \in y}((a, a) \in z)) \\ & \wedge (\forall_{a,b \in y}(((a, b) \in z \wedge a \neq b) \rightarrow (b, a) \notin z)) \\ & \wedge (\forall_{a,b,c \in y}(((a, b) \in z \wedge (b, c) \in z) \rightarrow (a, c) \in z)) \\ &)\end{aligned}$$

(vi) x is a toset

Answer. x is a toset iff x is a connected poset. So, we can expand the above definition of a poset with the following conjunction that states the ordering is connected:

$$\wedge\ (\forall_{a,b\in y}(a \neq b\ \rightarrow\ ((a,b) \in z\ \vee\ (b,a) \in z)))$$

(vii) x is a woset

Answer. x is a woset iff x is a well-founded toset. So, we again expand the above definition of a toset with the following conjunction that states the ordering is well-founded:

$$\wedge\ (\forall a((a \subseteq y\ \wedge\ a \neq \emptyset)\ \rightarrow\ \exists_{b\in a}\forall_{c\in a}(b \leq c))))$$

(viii) x is an ordinal

Answer. x is an ordinal iff $x = (y, z)$ is a woset such that $y_a = a$ for all a in y. We can further expand the definition of a woset with the following conjunction:

$$\wedge\ (\forall z(z \in y_a\ \leftrightarrow\ z \in a))$$

which is equivalent to...

$$\wedge\ (\forall z(z < a\ \leftrightarrow\ z \in a))$$

(ix) x and y are isomorphic wosets

Answer. $x = (i, u)$ and $y = (b, v)$ are isomorphic wosets iff x and y are wosets such that ...

(x) x is a group

(xi) x is an abelian group

Answer. Omitted since it requires a working knowledge of group-theory.

2.2 The Cumulative Hierarchy of Sets

Here, we develop further the idea of sets and, more specifically, a strict method for *constructing* sets. To illustrate why a formal method of constructing sets is important, let us start with a loose way of defining sets and see what trouble it leads us into.

From the last section, we saw how we can define a *collection* of sets by a formula in LAST. With this, it seems intuitive to define x as a set if and only if there is a formula $\phi(v_n)$ of LAST with just one free variable v_n where for every $a \in x$, $\phi(a)$. This intuitive way of defining sets, however, leads to an inconsistent set theory! To explain, let ϕ be the LAST formula

$$(\neg(v_0 \in v_0)) \tag{2.2.1}$$

So, if we follow our above definition, then ϕ defines a set x such that

$$x = \{a | a \notin a\} \tag{2.2.2}$$

Since x is a set, it is either the case that $x \in x$ or $x \notin x$. Let us consider both cases. (1) If $x \in x$ then x must satisfy ϕ, that is, $x \notin x$ *(a contradiction of our assumption!)* (2) If $x \notin x$ then x must fail to satisfy ϕ which means that $x \in x$ *(again, a contradiction of our assumption!)*

So, what was wrong with this method of defining sets? Simple: this method did not regard what was available in *constructing* our set! That is, when we were building the set x, we were presupposing that we already had x constructed (a sort of circular definition of x.)

Clearly, we need a more strict method of constructing and defining sets, and from the above argument we can begin to see the hierarchical nature of constructing sets.

For example, suppose our base set of objects which we can construct sets from are the numbers 0 and 1. From this, we could construct the sets $\emptyset$ (since the empty set is a subset of every set...a key point in a few minutes), {0}, {1}, and the set {0,1}. That's it. Those are all the sets we can construct. However, if we now extend our set of objects to include the sets we just constructed (remember, we can have sets of sets), we can then construct the following sets in addition to the previous sets: {0,{0}}, {0,{1}}, {1,{0}}, {1,{1}}, {{0},{1}}, {{0,1}}, and indeed a few more. We could then consider all of *these* sets *in addition* to the ones previously constructed to construct another level of sets.

Formally, then, where do we begin with this hierarchy of sets? What is our first 'level' of sets? and for how long can we extend this hierarchy of sets?

As stated earlier, the empty set is a subset of every set.

Proof. Remember, we defined $x \subseteq y$ as $(\forall v_n((v_n \in x) \ \rightarrow \ (v_n \in y)))$. So, for any set x, $\emptyset \subseteq x$ iff $(\forall v_n((v_n \in \emptyset) \ \rightarrow \ (v_n \in x)))$. Since the hypotheses $(v_n \in \emptyset)$ always fails, the statement is always true. □

With this, we start our hierarchy of sets with

$$V_0 = \emptyset \tag{2.2.3}$$

where V_0 denotes the first level of the set theory hierarchy. To make this hierarchy as powerful as possible, we place no restrictions on how many levels we allow in this hierarchy. So, for each *ordinal number* α, we define V_α as the set of elements in $\bigcup_{\beta<\alpha} V_\beta$.

Now, suppose we have V_α defined. What sets are in $V_{\alpha+1}$? Our intention is clear: $V_{\alpha+1}$ should consist of all possible sets definable in V_α, but how do we accurately define this? For now, we don't. As Devlin states, it will not be until Chapter 5 that we discuss the *Axiom of Constructibility*, which will allow us to define what we need. In the mean time, we accept without definition the notion of an *unrestricted power set* operation. We shall simply assume that given any set x, the power-set of x (abbreviated $\mathcal{P}(x)$) simply exists, and consists of all and only the subsets of x.

So, given our intention that $V_{\alpha+1}$ should consist of all possible sets definable in V_α, we can 'formally' define $V_{\alpha+1}$ as

$$V_{\alpha+1} = \mathcal{P}(V_\alpha) \tag{2.2.4}$$

This tells us how, given V_α, to construct $V_{\alpha+1}$, and since we defined V_0 to be $\emptyset$ we can easily construct $V_1, V_2, \ldots$

To summarize, we define the *cumulative hierarchy of sets* (a.k.a. the *Zermelo hierarchy*) as

$$\begin{array}{lcl} V_0 & = & \emptyset \\ V_{\alpha+1} & = & \mathcal{P}(V_\alpha) \\ V_\alpha & = & \bigcup_{\beta<\alpha} V_\beta, \text{if } \alpha \text{ is a limit ordinal} \end{array}$$

One last bit of theory covered in this section is the *Axiom of Subset Selection*. This axiom states that for any given set, there may be numerous subsets, but that every definable *collection* of subsets exists. Remember, we can define a *collection* of sets in LAST by using a formula with one free variable, written as $\phi(v_n)$. To remove any danger from defining sets in this fashion (remember the contradiction we reached earlier) we can limit ourselves That is, let x be any given set. The formula $\phi(v_n)$ applied to x then defines a subset of x in which every element a, $\phi(a)$ holds true.

Exercise 2.2.1. Show that if $y \in V_\alpha$ and $x \in y$, then $x \in V_\alpha$.

Proof. In the below proof, we can safely assume that $\alpha > 1$. Otherwise, the statement would be vacuously true (simple proof).
Definition: Let $B = \{\beta < \alpha : \exists_{x\in V_\beta}\exists_{y\in x}(y \notin V_\beta)\}$. Let β_0 be the minimum element of B.
Case 1: $\beta_0 = \emptyset$. If this is the case, then there are no members of β_0 (it's empty) so proving transitivity is trivial.
Case 2: β_0 is a successor ordinal. That is, $\beta_0 = \gamma + 1$ for some γ. Pick $x \in V_{\gamma+1} \ \wedge \ x \nsubseteq V_{\gamma+1}$ (from our assumption, we know such an x must exist.) Now, $x \subseteq V_\gamma$ since $V_{\gamma+1} = \mathcal{P}V_\gamma$. Since $x \nsubseteq V_{\gamma+1}$ there is a $y \in x$ where $y \notin V_{\gamma+1}$. Since $\gamma < \beta_0$ then V_γ *is* transitive! So, $x \subseteq V_\gamma$ and then $y \subseteq V_\gamma$. Thus, $y \in \mathcal{P}V_\gamma$, $y \in V_{\gamma+1}$ and $y \in V_{\beta_0}$. $\bot$!
Case 3: β_0 is a limit ordinal. Pick $x \in V_{\beta_0} \ \wedge \ x \nsubseteq V_{\beta_0}$. Then for some $\gamma < \beta_0$ we know $x \in V_\gamma$. Since $\gamma < \beta_0$, V_γ is transitive! Since $x \in V_\gamma$ then $x \subseteq V_\gamma$. Then $x \subseteq (V_\gamma \cup *)$, so $x \subseteq V_{\beta_0}$! $\bot$
Thus, the V_α hierarchy is transitive. □

Exercise 2.2.2. Show that, for any ordinal α, $V_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}(V_\beta)$.

Proof. Omitted. □

Exercise 2.2.3. Show that, if $\alpha < \beta$, then $V_\alpha \subset V_\beta$.

Proof. First, let's use the lemma that Professor Schlipf provided us in class.

1	$\alpha < \beta$	Premise
2	$V_\beta \bigcup_{\gamma<\beta} \mathcal{P}(V_\gamma)$	From exercise 2.2
3	$\mathcal{P}(V_\alpha) \subseteq V_\beta$	since $\alpha < \beta$ and above line
4	$V_\alpha \in \mathcal{P}(V_\gamma)$	since $V_\alpha \subseteq V_\alpha$
5	$V_\alpha \in V_\beta$	since $\in$ from above two lines

So, with this lemma, we know that if $\alpha < \beta$ that $V_\alpha \in V_\beta$. With this lemma, our proof becomes trivial. Note that $V_\alpha \subset V_\beta$ is the same as $V_\alpha \subseteq V_\beta \ \wedge \ V_\alpha \neq V_\beta$.

1	$\alpha < \beta$	Premise
2	$V_\alpha \in V_\beta$	From the above lemma
3	$V_\alpha \subseteq V_\beta$	From exercise 2.1
4	$\quad V_\alpha = V_\beta$	Premise
5	$\quad V_\alpha \in V_\alpha$	simple substitution with line 2
6	$\quad \bot$	since $x \notin x$!
7	$V_\alpha \neq V_\beta$	$\bot$-elimination
8	$V_\alpha \subseteq V_\beta \ \wedge \ V_\alpha \neq V_\beta$	$\wedge$ -introduction
9	$V_\alpha \subset V_\beta$	

□

Exercise 2.2.4. Check the following:

(i) $V_0 = \emptyset$

Answer. Trivial... V_0 is defined to be $\emptyset$

(ii) $V_1 = \{\emptyset\}$

Answer. By definition, V_1 is defined to be $\mathcal{P}(V_0)$. The $\mathcal{P}(V_0)$ is the set containing all subsets of V_0. The *only* subset of V_0 is, indeed, the set $\emptyset$. So, for V_1, the only set we can 'construct' is the set containing $\emptyset$.

(iii) $V_2 = \{\emptyset, \{\emptyset\}\}$

Answer. Using a similar argument from part (i), we see immediately that all elements in the sets V_0 and V_1 are in V_2.

(iv) $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

Answer. Similar to the last two levels, we see we have the union of all sets from the previous levels of the hierarchy. Also, we see the addition of the set $\emptyset$, which does not appear in any of the previous levels. It appears because the set $\emptyset$ was in a previous level (V_2), and the *subset* containing only this set is $\emptyset$.

Exercise 2.2.5. What are V_4 and V_5?

Answer.

$$
\begin{aligned}
V_4 \ = \{ \ & \emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\{\{\emptyset\}\}\},\ \{\emptyset,\{\emptyset\}\},\ \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}, \\
& \{\emptyset,\{\{\emptyset\}\}\},\ \{\emptyset,\{\emptyset,\{\emptyset\}\}\},\ \{\{\emptyset\},\{\{\emptyset\}\}\},\ \{\{\emptyset\},\{\emptyset,\{\emptyset\}\}\},\ \{\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}, \\
& \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\},\ \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\},\ \{\emptyset,\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\},\ \{\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\} \\
\} &
\end{aligned}
$$

whew!
Notice how the first line has all 1-element subsets of $\mathcal{P}(V_3)$, the second line has all 2-element subsets of $\mathcal{P}(V_3)$, and the last line has all 3-element subsets of $\mathcal{P}(V_3)$.
As for V_5, I leave its construction to anyone who wishes to write all 65,536 elements (*without* the use of a computer!)

Exercise 2.2.6. How many elements has V_n, where n is a positive integer?

Answer. Before we provide a general answer, notice the trend so far...

Set	Number of Elements
V_0	0
V_1	1
V_2	2
V_3	4
V_4	16

Just by looking at this trend, we immediately see that the number of elements for each of the sets (except V_0) are perfect powers of two. As it turns out, for a set v_n of size x, the $\mathcal{P}(v_n)$ has precisely 2^x elements. We adopt the syntax $|x|$ to denote the number of elements in a set x. With that, we can generalize the trend as follows:

Set	Number of Elements
V_0	0
V_{n+1}	$2^{\lvert V_n \rvert}$

2.3 The Zermelo-Fraenkel Axioms

Until now, we have made certain *assumptions* regarding sets and we are not ready to remove these assumptions by constructing an axiom base for set theory. We want to include in our axioms *only* what we needed to construct the V_α-hierarchy of sets discussed in the last section.

The first bit of information we required in constructing the V_α- hierarchy was the ordinal numbering system. So, we take the ordinal numbering system as basic; that is, we just accept the existence of ordinal numbers without proof.

Next, we look at power-set operation, which is what we used to construct the consecutive levels of the V_α-hierarchy. We create the *Power Set Axiom*:

. *Power Set Axiom. If x is a set, there is a set that consists of all and only the subsets of x.*

Exercise 2.3.1. Write down a LAST sentence which expresses the power set axiom.

Answer. For every set x, we can define the $\mathcal{P}(x)$ as follows:

$$\forall x \exists y \forall z (z \subseteq x \leftrightarrow z \in y) \tag{2.3.1}$$

The power set axiom does not yet allow us to construct the entire V_α- hierarchy. What if α is a limit ordinal? In that case, our definition of $V_{\alpha+1}$ required the *union* of a collection of sets. So, we create the *Axiom of Union*:

. *Axiom of Union. If x is a set, there is a set whose members are precisely the members of the members of x, i.e. the set $\bigcup x$.*

Exercise 2.3.2. Express the axiom of union in LAST.

Answer. We can express y as the $\bigcup x$ as follows:

$$\forall x \exists y \forall z (z \in x \leftrightarrow \forall a (a \in z \leftrightarrow a \in y)) \tag{2.3.2}$$

The next axiom is stressed as the most important, but least appreciated. It allows us to construct a set x, from another set y, by replacing every element a in x with a new set a'. We allow any replacement that is definable by a LAST formula.

. *Axiom of Replacement. Let $\phi(v_n, v_m)$ be any formula of LAST such that for each set a there is a unique set b such that $\phi(a, b)$. Let x be a set. Then there is a set y consisting of just those b such that $\phi(a, b)$ for some a in x.*

Exercise 2.3.3. In LAST, we do not have the ability to work with whole formulas of LAST, so there is no way to translate the Axiom of Replacement into a LAST formula. Instead, however, we can construct a formula schema. Given any formula $\phi(v_n, v_m)$ of LAST write down a sentence of LAST that expresses the Axiom of Replacement.

Answer.

$$\forall d \exists r \forall y (y \in r \leftrightarrow \exists x (\phi(x, y) \wedge x \in d)) \tag{2.3.3}$$

The next two axioms are the last two we need to construct the cumulative hierarchy of sets. The first axiom simply asserts the existence of the empty set. The second axiom is used for the construction of ordinals. Other axioms, such as the Powerset axiom, require the existence of sets. These next two axioms (combined with the others) allow us to actually *construct* sets.

. *Null Set Axiom. There is a set which has no members (denoted by $\emptyset$.)*

. *Axiom of Infinity. There is a set x such that $\emptyset \in x$, and such that $\{a\} \in x$ whenever $a \in x$.*

The Axiom of Infinity guarantees the existence of *at least one* set. We could, then, use the Axiom of Infinity along with the Axiom of Subset Selection, to prove the existence of the Null set, thus removing the need for the Null Set Axiom. For example, given some set a, we could express the Null set as:

$$\emptyset = \{x \in a | x \neq x\} \tag{2.3.4}$$

Exercise 2.3.4. Express the *Null Set Axiom* and the *Axiom of Infinity* in LAST.

Answer. For the *Null Set Axiom*, we can express $\emptyset$ as the set y such that y has no elements:

$$\exists y \forall x (\neg(x \in y)) \tag{2.3.5}$$

For the *Axiom of Infinity*, we define x as

$$\exists x (\emptyset \in x \ \wedge \ \forall a (a \in x \ \leftrightarrow \ \{a\} \in x)) \tag{2.3.6}$$

The next axiom is so fundamental that we accepted it in §1.1. It is the Axiom of Extensionality, and it allows to talk about two sets being "equal". As we did in §1.1, we define two sets to be equal when their elements are identical.

. *Axiom of Extensionality. If two sets have identical elements, then they are equal.*

Exercise 2.3.5. Express the *Axiom of Extensionality* in LAST.

Answer. Two sets, x and y, are defined to be equal if and only if:

$$\forall z (z \in x \ \leftrightarrow \ z \in y) \tag{2.3.7}$$

Using the axioms we have defined thus far, all that is needed to construct the cumulative hierarchy of sets is the fact that the membership operator ($\in$) be well-founded. So, we create the following axiom:

. *Axiom of Foundation. $\in$ is a well-founded relation. That is, for every nonempty set x, there is a set $a \in x$ such that $a \cap x = \emptyset$.*

It should be noted that the Axiom of Foundation is also referred to as the Axiom of Regularity.

Exercise 2.3.6. Prove that the relation $\in$ is well-founded if and only if for every nonempty set x, there is a set $a \in x$ such that $a \cap x = \emptyset$.

Answer. Is $\in$ a well-founded relation? Remember from page 11 of our text that a poset is said to be well-founded if every nonempty subset has a minimal element. So, this is equivalent to asking if for every set x, is $(x, \in)$ a well-founded poset? Look at the definition we provided earlier of a poset in exercise 2.1.1 (part v). If we use $\in$ as our relation, it becomes immediately apparent that $\in$ is *indeed* well-founded. *NOTE: We are actually defining a weak-poset, that is, one that is anti-reflexive. Remember that reflexivity is required for a normal poset, and $\in$ is clearly not reflexive, so it could not be a normal poset.*

Exercise 2.3.7. Assume axioms 1-7 (listed below) and prove that the Axiom of Foundation is equivalent to the equality:

$$V = \bigcup_{\alpha} V_{\alpha} \tag{2.3.8}$$

Answer. Omitted.

In finishing, the Zermelo-Fraenkel axioms are:

1. Axiom of Extensionality.
2. Null Set Axiom.
3. Axiom of Infinity.
4. Power Set Axiom.
5. Axiom of Union.
6. Axiom of Replacement.
7. Axiom of Subset Selection.
8. Axiom of Foundation.
9. Axiom of Choice (discussed in §2.7.)

NOTE: The theory consisting of Axioms 1-8 is called ZF set theory. If we add Axiom 9, the resulting theory is called ZFC set theory.

2.4 Classes

It should be noted the objections Devlin has with class theory. First, class theory looses the 'intuition' present in ZF set theory. Second, class theory is not needed. It can be proven formally that any result about sets provable in class theory is already provable in ZF set theory. As such, Devlin spends only a small amount of time introducing the basics of class theory.

We now have a good understanding of what we allow to be a set and what we do not. We allow the ordinals and anything in the V_α hierarchy to exist as sets, but nothing else. We have, however, used the term "collection" often, and as we have seen, collection can be determined by formulas of LAST (but not all collections can!)

So, we now extend our axiom system to handle 'collections', thus leaving the bounds of *set theory* and entering *class theory*. The basics of class theory are as follows:

1. All sets are classes.
2. Classes that are not sets are *proper classes*.

How can we have classes that are not sets? Some basic examples are the collection of all ordinals and V. Since these are not sets, we must not handle them in any way that we would a class. Even the basic idea that we constructed ZF set theory on is useless. . . it is meaningless for one collection to be a member of another collection.

Classes do, however, exhibit properties similar to sets. For example, consider the following classes:

$$A = \{\, x \mid \phi(x)\}$$
$$B = \{\, x \mid \psi(x)\}$$

In this case, we can express simple notions such as

$$A \subseteq B$$

with

$$\forall x(\phi(x) \ \rightarrow \ \psi(x))$$

Exercise 2.4.1. Let $A = \{x \mid \phi(x)\}$ and $B = \{x \mid \psi(x)\}$. Let C be the class $A \cup B$. Express the assertion $x \in C$ as a sentence in set theory.

Answer.

$$\phi(x) \ \vee \ \psi(x) \tag{2.4.1}$$

Exercise 2.4.2. Express the assertion $C = A \cap B$ as a sentence in set theory.

Answer.

$$\phi(x) \ \wedge \ \psi(x) \tag{2.4.2}$$

So, what *is* the benefit of studying classes? Simple: they help make some of the more complicated notions of set theory simpler. Look at the Axiom of Replacement, given last section. In class theory, the same notion can be replaced with the following:
Define the class F as follows:

$$F = \{(a, b)|\phi(a, b)\} \tag{2.4.3}$$

Notice that the class F looks just like a function in sets. Now, for any set x, the class:

$$\{\, F(a) \mid a \in x\} \tag{2.4.4}$$

is a set. And that is the Axiom of Replacement in class theory.

To summarize class theory:

1. Classes are just abbreviations for defining sets using formulas of LAST.
2. Proper classes are 'big collections' that are *not* sets.
3. Proper classes can be handled similar to sets, except that they are not a completed whole.
4. All sets are classes, but not all classes are sets.

Exercise 2.4.3. Let On be the class of all ordinals. Prove that On is a proper class.

Answer. The proof follows the outline given in the book. We show that if On is the *set* of all ordinals, then it too is an ordinal. With that, we would have $On \in On$ which violates the axiom of foundation (how?).

Proof. Suppose not. That is, suppose On is not a proper class and is indeed a set. On page 18 of our text, we define x to be an ordinal iff x is well-ordered and $X_a = a$ for all $a \in X$. We want to show that On is an ordinal. To do this, we have to prove that On is reflexive, antisymmetric, transitive, connected, and well-founded. We omit the proofs of all of these except well-foundedness since they are trivial (they are simple applications of their definitions which we provided in exercise 2.1.1.) So, let us prove that On is well-founded...

Let $z \subseteq On$ and $z \neq \emptyset$. Assume z does *not* have a minimum element, that is, there is no $a \in z$ such that $\forall_{b \in z}(a \not< z)$. Pick some $a \in z$. We know a is an ordinal.
Case 1: $a \cap z$ is empty. Then a is less than all elements of z! That is, a is the minimum element of z, which contradicts our assumption about z.
Case 2: $a \cap z$ is not empty. Then $a \cap z$ is the set of all ordinals in z that are $< a$. So, $a \cap z$ has a minimum element. Let b be the minimum element. Now we prove that b is the minimum element of z. Pick some other element $c \in z$.
Case 2.1: $c \in a$. Then, $b < c$ by our choice of b.
Case 2.2: $c \notin a$. Then $c \geq a$ since $b \subseteq c$. So, $b < c$.
So, z does have a minimum element! And On is well-founded. Well-foundedness together with the other properties listed above imply that On is itself an ordinal. Thus, $On \in On$, which is a contradiction.
Thus, On *is* a proper class. □