Mathematical Logic III
The Joy of Sets
Chapter 5
Ryan Flannery

5 The Axiom of Constructibility

In Chapter 2, we defined the V hierarchy of sets making use of what we called the *unrestricted power set operation*. The power set axiom allowed us to keep our notions of *set* and *subset* vague.

In this chapter, we cover an *extension* to ZF set theory that attempts to remove this vagueness.

5.1 Constructible Sets

Let a set be any *describable* collection. How do we "describe" a set? By the only means we have: LAST. Thus, any collection that can be described by a formula of LAST is a set. Using this new definition, we can redefine the set-theoretic hierarchy.

In the V hierarchy, we made use of the *unrestricted power set* operation. In the new hierarchy, we introduce the *describable power set* operation.

Unrestricted Power Set The unrestricted power set of X is the set of *all* subsets of X.

Describable Power Set The describable power set of X is the set of all *describable* (by means of LAST) subsets of X.

Precise Definition: $X \in L_{\alpha+1}$ if and only if there is a formula $\phi(v_n)$ of LAST, with the single free variable v_n, and sets $a_1, \ldots, a_m$ in L_α, which interpret the names involved in ϕ, such that X is the collection of all $x \in L_\alpha$ for which $\phi(x)$ is true. Any quantifiers in ϕ are interpreted over L_α since the only sets we can describe (at stage α) exist in L_α. The describable power set is the set of all such X.

Just as we did in the V hierarchy, we begin the hierarchy with the empty set and at limit levels we take the union of all previous levels.

$$L_0 = \emptyset \tag{5.1.1}$$

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \tag{5.1.2}$$

$$L_{\alpha+1} = \text{The describably power set of } L_\alpha \tag{5.1.3}$$

With that, our constructible universe is:

$$L = \bigcup_{\alpha} L_\alpha \tag{5.1.4}$$

5.2 The Constructible Hierarchy

The following properties of the V hierarchy are also shared with the constructible hierarchy:

- $L_\alpha \subseteq L_\beta$ for $\alpha \leq \beta$
- each L_α is transitive ($x \in L_\alpha \ \rightarrow \ x \subseteq L_\alpha$)
- $L_\alpha \cap On = \{\beta \mid \beta < \alpha\} = \alpha$

This is, however, where the similarities end. The following are some of the key differences between the L hierarchy and the V hierarchy.

- $|L_\alpha| = |\alpha|$ for every infinite ordinal α. This is a direct consequence of LAST being countable.
- With that, $|L_{\omega+1}| = \aleph_0$. Trivial, since $\aleph_0 = |\omega + 1|$.
- However, since $\mathcal{P}(\omega) \subseteq V_{\omega+1}$, we have that $|V_{\omega+1}| > \aleph_0$.
- So, $L_{\omega+1} \neq V_{\omega+1}$! This is where the L hierarchy begins to separate from the V hierarchy. The L hierarchy grows *much* more slowly than the V hierarchy.

- *Hold on!* From what we stated above, $L_{\omega+1}$ is countable. We proved earlier, however, that $\mathcal{P}(\omega)$ is uncountable. How is this possible if $\mathcal{P}(\omega) \subseteq L_{\omega+1}$? Simple: not all of $\mathcal{P}(\omega)$ will be contained in $L_{\omega+1}$. Remember, we only include those subsets of ω that are describable by LAST, and $\mathcal{P}(\omega)$ contains *all* subsets (describably or not) of ω.

5.3 The Axiom of Constructibility

Now, just as we did in §2.2, we look at the L hierarchy and isolate all those assumptions we made that were necessary in constructing L. These assumptions will be our axioms for this constructible set theory. As it turns out, the assumptions we made were a subset of those assumptions we made when constructing the V hierarchy. As such, the ZF axioms suffice for building the constructible hierarchy! We want to go a step further, however, and show that *the* universe is constructible. That is, that $V = L$. This requires an additional axiom that is called the *Axiom of Constructibility.* So we can summarize constructible set theory as:

1. The ZF axioms
2. $V = L = \bigcup_\alpha L_\alpha$ - The Axiom of Constructibility

So, constructible set theory is an *extension* of ZF set theory, and is typically denoted as $ZF + (V = L)$
Key consequence of $ZF + (V = L)$: AC is provable!

Theorem 5.3.1. *In $ZF + (V = L)$, every set can be well-ordered.*

Proof. As Devlin states, the proof of this theorem (done by Gödel) is beyond the scope of this book. Page 126 of the text provides a brief (very brief) overview of the proof. □

So, the axiom of constructibility is equivalent to fixing a precise definition of what a set is!

Notation Considerations From now own, we always take the ZF axioms to be basic. If any theorem requires AC, it should explicitly state it. Similarly, if any theorem requires $V = L$, it should explicitly state it.

5.4 The Consistency of $V = L$

$ZF + (V = L)$ seems nice, but is it consistent? We learned last quarter that within any system, the consistency of the system can not be proven. And since we are adding an additional axiom to ZF (one that isn't intuitive), $ZF + (V = L)$ may seem more susceptible to inconsistency. Fortunately, Gödel showed otherwise.

Theorem 5.4.1. *If ZF is consistent, so to is $ZF + (V = L)$.*

Proof. Once again, Devlin states that the proof of this theorem is beyond the scope of this book. The basic approach of the proof is this: Gödel shows that *any* model for ZF is also a model for $ZF + (V = L)$. □

Since $ZF + (V = L)$ proves AC, we have the following corollary:

Corollary 5.4.1. *If ZF is consistent, so too is ZFC.*

5.5 Uses of the Axiom of Constructibility

Here, we look at some of the things that the axiom of constructibility affords us.

- Assuming $V = L$, AC is provable (as we saw in §5.3.)
- Assuming $V = L$, GCH holds.

  Proof. Alas! Yet again Devlin informs us that this proof is beyond the scope of this book. □

- Assuming $V = L$, the combinatorial consequence known as $\diamond$ holds.

  > There is a sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that for each $\alpha < \omega_1$, $S_\alpha \subseteq \alpha$, and whenever $X \subseteq \omega_1$, then for some infinite ordinal $\alpha \in \omega_1$, $X \cap \alpha = S_\alpha$.

  How $V = L$ implies $\diamond$ is, you guessed it, beyond the scope of this book. However, $\diamond$ implies CH.

In general, although $V = L$ is not intuitive the way most of the other ZF axioms are, assuming it affords us proofs for CH/GCH, AC, $\diamond$, and the Souslin Problem (briefly mentioned at the beginning of §5.1). Further, as Gödel proved, if ZF is consistent (which we have no way of knowing) then so is $ZF + (V = L)$. So $V = L$ is a "safe" extension of ZF.