Mathematical Logic I notes; April 4, 2005

Completeness Theorem for Propositional Logic

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1. **Lemma:** *Let A be a set of formulas. For each $\phi \in A$, $A \models \phi$.*

   Proof: We need to show that the following holds for every truth assignment ν: if $\nu \models \psi$ for every $\psi \in A$, then $\nu \models \phi$. But this is trivial: if $\nu \models \psi$ for every $\psi \in A$, and since $\phi \in A$, then $\nu \models \phi$ by assumption.

2. **Lemma:** *Let A be a set of formulas. For each $\phi \in A$, $A \vdash \phi_i$.*

   Proof: Fix any $\phi \in A$. The following is a natural deduction proof of ϕ from premises A:

   1. ϕ premise

3. **Lemma:** *Let ϕ be a formula and A, B be sets of formulas. If $A \subseteq B$ and $A \models \phi$, then $B \models \phi$.*

   Proof: We need to show that, for every truth assignment ν, if $\nu \models B$, then $\nu \models \phi$. So let ν be any truth assignment where $\nu \models B$, i.e., $nu \models \psi$ for every ψ in B. But then, since $A \subseteq B$, $nu \models \psi$ for every ψ in A, i.e., $\nu \models A$. So, by hypothesis, $\nu \models \phi$, as desired.

4. **Lemma:** *Let ϕ be a formula and A, B be sets of formulas. If $A \subseteq B$ and $A \vdash \phi$, then $B \vdash \phi$.*

   Proof: Let P be any proof of ϕ from premises A. That means that every premise in P is an element of A. But then every premise in P is also in B, so P is also a proof of ϕ from premises B.

5. **Lemma:** *Let ϕ, ψ be formulas and A be a set of formulas. $A \models (\phi \to \psi)$ if and only if $A \cup \{\phi\} \models \psi$.*

   Proof: We need to prove both directions of the "if and only if."

   "If": *We are given that $A \cup \{\phi\} \models \psi$, and we need to prove that $A \models (\phi \to \psi)$.* That is, we need to show that for any truth assignment ν, if $\nu \models A$, then $\nu \models (\phi \to \psi)$.

   Let ν by any truth assignment where $\nu \models A$. If $\nu \not\models \phi$, then $\nu \models (\phi \to \psi)$ (vacuously). On the other hand, if $\nu \models \phi$, then $\nu \models A \cup \{\phi\}$, so, by hypothesis, $\nu \models \psi$, and thus $\nu \models (\phi \to \psi)$. Thus, in any case, $\nu \models (\phi \to \psi)$, as desired.

   "Only if": *We are given that $A \models (\phi \to \psi)$, and we need to prove that $A \cup \{\phi\} \models \psi$.* That is we need to show that, for any truth assignment ν, if $\nu \models A \cup \{\phi\}$, then $\nu \models \psi$.

   Let ν be any truth assignment where $\nu \models A \cup \{\phi\}$. Then $\nu \models A$ also. Hence, by hypothesis, $\nu \models (\phi \to \psi)$. But, by the "truth table" for $\to$, since $\nu \models \phi$ and $\nu \models (\phi \to \psi)$, it must be that $\nu \models \psi$.

6. **Lemma:** *Let ϕ be a formula and A be a set of formulas. $A \models (\neg\phi)$ if and only if $A \cup \{\phi\}$ is unsatisfiable.*

   Proof: This follows from Lemma 5 with $\psi = \bot$ — for you can easily show that $\neg\phi$ is logically equivalent to $(\phi \to \bot)$, and $A \cup \{\phi\}$ is unsatisfiable iff $A \cup \{\phi\} \models \bot$.

7. **Lemma:** *Let $\phi_1, \phi_2, \ldots, \phi_k, \psi$ be formulas and A be a set of formulas. Then*

   $$A \models (\phi_1 \to (\phi_2 \to (\cdots \to (\phi_k \to \psi)) \cdots))) \text{ if and only if } A \cup \{\phi_1, \phi_2, \ldots, \phi_k\} \models \psi.$$

   Proof: By (ordinary) induction on k. The base case, the result for $k = 0$, is trivial. The inductive step is just an instance of Lemma 5.

8. **Lemma:** *Let ϕ, ψ be formulas and A be a set of formulas. $A \vdash (\phi \to \psi)$ if and only if $A \cup \{\phi\} \vdash \psi$.*

   Proof: We need to prove both directions of the "if and only if." Note that here we show how to manipulate proofs: how to convert a proof of one result into a proof or the other.

   "Only if" direction: *We need to show that if $A \vdash (\phi \to \psi)$ then $A \cup \{\phi\} \vdash \psi$.* So suppose the former. Let P be a natural deduction proof of $(\phi \to \psi)$ from A. Let k be the line number of the last line of P — so that the formula appearing in line k is $(\phi \to \psi)$.

So the following is a proof of ψ from $A \cup \{\phi\}$:

$$\begin{array}{lll} & & P \\ k+1. & \phi & \text{premise} \\ k+2. & \psi & \rightarrow\text{-elim. } k,\ k+1. \end{array}$$

For you can check that P itself obeys all the rules of being a natural deduction proof and that the two additional steps keep the proof legal.

"If" direction: *We need to show that if $A \cup \{\phi\} \vdash \psi$ then $A \vdash (\phi \rightarrow \psi)$.* So suppose the former. Let P be a natural deduction proof of $A \cup \{\phi\} \vdash \psi$.

Let k be the number of steps in P, and form P' be P as follows: (i) delete the line numbers on the lines, (ii) whenever ϕ appears in P, but not inside a subproof, with justification "premise", change that justification to "copy 1", and (iii) add 1 to all the line numbers used in justification within P.

The formula on the last line of P must be ψ, and that cannot be inside any subproofs. So the following is a proof of $(\phi \rightarrow \psi)$ from A:

$$\begin{array}{ll|lc|} 1. & & \phi & \text{assumption} \\ 2. & & & \vdots \\ \vdots & & & P' \\ k+1. & & & \vdots \\ k+2. & (\phi \rightarrow \psi) & & \rightarrow\text{-intro. } 2 \text{ - } k+1. \end{array}$$

There is some fussy checking involved to making sure that the result is indeed the proof I claim it is — but I leave all that checking to you.

9. **Lemma:** *Let ϕ be a formula and A be a set of formulas. $A \vdash (\neg\phi)$ if and only if $A \cup \{\phi\} \vdash \bot$.*

   Proof: Like the proof of Lemma 6, with Lemma 8 replacing Lemma 5.

10. **Lemma:** *Let $\phi_1, \phi_2, \ldots, \phi_k, \psi$ be formulas and A be a set of formulas. Then*

$$A \vdash (\phi_1 \rightarrow (\phi_2 \rightarrow (\cdots \rightarrow (\phi_k \rightarrow \psi))\cdots))) \text{ if and only if } A \cup \{\phi_1, \phi_2, \ldots, \phi_k\} \vdash \psi.$$

   Proof: Like the proof of Lemma 7, with Lemma 8 replacing Lemma 5.

11. **Lemma:** *Let A be any set of formulas and ϕ, ψ be any formulas. Then:*

    (a) *If $A \vdash \phi$ and $A \vdash \psi$, then $A \vdash (\phi \wedge \psi)$.*
    (b) *If $A \vdash \neg\phi$ or $A \vdash \neg\psi$, then $A \vdash \neg(\phi \wedge \psi)$.*
    (c) *If $A \vdash \phi$ or $A \vdash \psi$, then $A \vdash (\phi \vee \psi)$*
    (d) *If $A \vdash \neg\phi$ and $A \vdash \neg\psi$, then $A \vdash \neg(\phi \vee \psi)$*
    (e) *If $A \vdash \neg\phi$ or $A \vdash \psi$ then $A \vdash (\phi \rightarrow \psi)$.*
    (f) *If $A \vdash \phi$ and $A \vdash \neg\psi$, then $A \vdash \neg(\phi \rightarrow \psi)$.*

    Proof: All parts can be proved similarly to Lemma 8.

12. **N.B.** Nowhere above did we assume that A is finite. The *only* place so far that we have used the finiteness of A is in asserting that it is decidable, given a candidate proof P, whether P really is a proof from premises A.

    Defn: A set A of formulas is *inconsistent* if $A \vdash \bot$. (By comparison, A is *unsatisfiable* if $A \models \bot$. These two are equivalent — *but* we haven't prove that yet.)

    Lemma: *Suppose a set of formula A is inconsistent. Then some finite $A_0 \subseteq A$ is inconsistent.*

    Proof: Let P be a proof of $\bot$ from A. Since P is a finite sequence of numbered lines (ending with a line with formula $\bot$), only finitely many formulas in A can appear in P as premises. Let A_0 be exactly that set of formula in A.